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Toeplitz operators with analytic symbols and corona problems

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Aos meus pais

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Toeplitz operators with analytic symbols and corona problems

Abstract

The study of the properties of a Toeplitz operator T_G with 2×2 symbol G is related with an appropriate factorization of G. In particular, T_G is Fredholm if and only if G admits a Wiener-Hopf (WH) factorization and it is invertible if and only if this factorization is canonical. For almost periodic (AP) symbols, the so-called AP factorization appears as a natural generalization of the WH factorization, which does not exist for such matrices unless it is canonical. The existence and the actual determination of those factorizations are shown to be closely related to certain corona problems whose data are particular solutions to a Riemann-Hilbert problem $Gh_+ = h_-$, $h_\pm \in (H_\infty^\pm)^2$.

In this thesis, on the one hand, conditions are established which are equivalent to the corona conditions but easier to verify, if $G^{\pm 1}$ are analytic and bounded in a strip of the complex plane. On the other and, new classes of symbols G, for which a non-trivial solution to the Riemann-Hilbert problem can be explicitly determined and the corona conditions can be verified by the above mentioned approach, are identified. Criteria for factorability of matrix function G in those classes are thus obtained.

Keywords. Toeplitz operator; Riemann-Hilbert problem; corona theorem; Wiener-Hopf factorization; AP factorization

Operadores de Toeplitz com símbolos analíticos e problemas da coroa

Resumo

O estudo das propriedades de um operador de Toeplitz T_G com um símbolo matricial 2×2, G, está relacionado com uma factorização apropriada de G. Em particular, T_G é de Fredholm se e só se G admite uma factorização de Wiener-Hopf (WH) e é invertível se e só se essa factorização é canónica. Para símbolos quase-periódicos (AP), a chamada factorização AP aparece como uma generalização natural da factorização WH, a qual não existe para essas matrizes a menos que seja canónica. Mostra-se que a existência e a determinação dessas factorizações estão intimamente relacionadas com certos problemas da coroa cujos dados são soluções particulares do problema de Riemann-Hilbert $Gh_+ = h_-$, $h_\pm \in (H_\infty^\pm)^2$.

Nesta tese, por um lado, são estabelecidas condições que são equivalentes às condições da coroa mas mais fáceis de verificar, se $G^{\pm 1}$ forem funções matriciais analíticas e limitadas numa faixa do plano complexo. Por outro lado, são identificadas novas classes de símbolos G para as quais uma solução não trivial do problema de Riemann-Hilbert pode ser determinada explicitamente e as condições da coroa podem ser verificadas pela abordagem anterior. São assim obtidos critérios de factorização de funções matriciais G nessas classes.

Palavras-chave. Operador de Toeplitz; problema de Riemann-Hilbert; teorema da coroa; factorização de Wiener-Hopf; factorização AP

Contents

Chapter 1

Introduction

Toeplitz operators play an important role in many areas of Mathematics, Physics and Engineering, which largely explains the continuing and even growing interest in the study of their properties, namely invertibility, Fredholmness, dimension of their kernel and codimension of their image. There is an enormous literature devoted to this topic; see, for instance, the references in [8].

Moreover, the relation between the properties of a Toeplitz operator T_G and the factorization of its symbol G (cf. [8], [16], [24]) is one reason explaining why a lot of effort has been done to develop new criteria of existence of such a factorization, as well as new methods for explicitly obtaining it.

Since there is no general answer to this problem, the approach to its study depends mostly on the type of symbol associated with the Toeplitz operator. For instance, conditions for existence of a Wiener-Hopf factorization for a class of matrix functions which include the piecewise continuous functions in $\dot{\mathbb{R}}$ are known (cf. [16], [23] and [24]); the main questions that remain concern therefore the determination of the factors.

Nevertheless, some other classes of symbols have attracted particular attention in the literature. This is the case, for instance, of Toeplitz operators with oscillatory symbols, intimately connected with finite interval convolution operators, which are an important class presenting great difficulties (cf. [3, 4, 5, 7, 8, 9, 12, 13, 14, 15, 20, 21, 25, 26, 27, 28, 29]).

It should be noticed that, in the last years, new approaches have allowed to develop this study. Some relevant examples are the use of the corona theorem, which is of great value from the point of view of Complex Analysis as well as from that of Operator Algebras, as a tool to determine conditions for invertibility of Toeplitz operators (and even expressions for the inverse operator, if two associated corona problems can be solved), the development of methods to solve some particular Riemann-Hilbert problems and, in the case of almost periodic (AP) factorization, the Portuguese Transformation approach (see Chapter 5).

The motivation for the work presented in this thesis came from the study of problems leading to finite interval convolution equations. Toeplitz operators with AP symbols naturally appear in this setting. It happens that those matrix functions present some interesting particularities. In fact, on the one hand, for these functions it often happens that their behaviour "at infinity" and in a strip of the complex plane can be studied separately. On the other hand, for this class of matrix symbols, Fredholmness of Toeplitz operators is equivalent to its invertibility. Therefore, if a Wiener-Hopf factorization exists, then it is canonical and it coincides with a canonical APW factorization of G when $G \in (APW)^{2\times 2}$ (cf. [7]).

However, many results of this thesis are applicable in the setting of matrix functions which are not necessarily almost periodic. Therefore we start by focusing our study on the invertibility of operators with analytic symbols, which are not necessarily AP, and, to a smaller extent, its Fredholmness.

When the Toepliz operator is not invertible (nor Fredholm), but the symbol G is an APW matrix function, then we study the existence of an (non-canonical) APW factorization of G.

We now outline the structure of this thesis. Chapter 2 is a review of some definitions and known results that we frequently use throughout this thesis.

Chapter 3 is divided in three sections. In Section 3.1, we start by reviewing the corona theorem and the importance of the Riemann-Hilbert problem

$$G\phi_+ = \phi_- \,, \qquad \phi_\pm \in \left(H_\infty^\pm\right)^2 \,, \tag{1.1}$$

when we consider the question of existence and determination of a canonical Wiener-Hopf factorization for a 2×2 matrix function G with entries in $L_\infty(\mathbb{R})$, based on an important result of [2].

We show that not only can a solution to (1.1) give us important information about the existence of a canonical Wiener-Hopf factorization of $G \in (L_\infty(\mathbb{R}))^{2\times 2}$, but also about an AP factorization when $G \in (APW)^{2\times 2}$. Therefore, for $G \in (APW)^{2\times 2}$, we also address the problem of what conclusions can be taken from this solution as regards the existence of an APW factorization of G and, if the latter exists, we give formulas for its AP partial indices and factors.

On the other hand, these results raise other questions because not only is there no general method to determine a particular solution to the associated Riemann-Hilbert problem, but it is also, in general, very difficult to verify whether or not the corona conditions are satisfied for that particular solution. Thus, in Section 3.2, we establish conditions on the solutions to (1.1), $\phi_\pm$, which are equivalent to, or imply the corona conditions, but are simpler to verify, by taking advantage of some properties of G. In fact, to verify that $\phi_\pm \in CP^\pm$, we must show that ϕ_{1+} and ϕ_{2+} do not approach 0 simultaneously in the upper half-plane and analogously for ϕ_{1-} and ϕ_{2-} in the lower half-plane. This is generally difficult or even impossible to do directly, given the usually complicated expressions of those functions. However it turns out that, in several important cases, it is easy to see whether ϕ_{1+}, ϕ_{2+} can approach 0 simultaneously in $\mathbb{C}^+ + i\varepsilon_1$ if $\varepsilon_1 > 0$ is big enough (and analogously for ϕ_{1-}, ϕ_{2-} in $\mathbb{C}^- - i\varepsilon_1$). Therefore, we are reduced, in those cases, to studying the behaviour of $\phi_{1\pm}$, $\phi_{2\pm}$ in a strip in the complex plane. So in this section we show that, if the elements of $G^{\pm 1}$ are analytic and bounded in a strip $S = \{z \in \mathbb{C} : \operatorname{Im} z \in \,] - \varepsilon_2, \varepsilon_1 [\, \}$ with $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$, then the conditions

(i) $\inf_S \left(|\phi_{1+}| + |\phi_{2+}| \right) > 0$,

(ii) $\inf_S \left(|\phi_{1-}| + |\phi_{2-}| \right) > 0$,

(iii) $\inf_S \left(|\phi_{2+}| + |\frac{1}{c}(\phi_{2-} - d\phi_{2+})| \right) > 0$,

(iv) $\inf_S \left(|\phi_{2-}| + |\frac{1}{c}(a\phi_{2-} - \gamma\phi_{2+})| \right) > 0$,

where $\gamma = \det G$ and a, d, c are elements of G, are equivalent. This means that, as far as approaching 0 simultaneously in S is concerned, we are free to choose a particular pair of functions which is easier to study (for instance, only ϕ_{1+} and ϕ_{2+}).

It should be noticed that several important classes of symbols present the above mentioned property of analyticity in a strip. This is the case of almost periodic symbols, which have attracted great attention in mathematical publications ([9, 13, 14, 15, 20, 25, 26, 28, 29], see [7] for more references).

If the solutions to (1.1) are almost periodic polynomials, then it is easy to know their behaviour "at infinity". In Section 3.3 we show that, through an appropriate change of variables, we may be able to reduce the verification of the corona conditions in a strip to the study of the common zeros of two polynomials. An interesting aspect here is the interplay between classical Real Analysis and Functional Analysis which is put in evidence in the way these results are proved.

It is clear that, as it happens with many other methods, the one proposed here cannot be always applied. Nevertheless, we will show its usefulness by applying it, in a clear and relatively simple way, to the study of the factorability of several kinds of matrix functions. So in Chapter 4 we identify new classes of symbols G for which a non-trivial solution to (1.1) can explicitly be obtained and the corona conditions can be verified. For all classes of matrix functions studied in this chapter, G takes the form

$$G = \begin{bmatrix} e_{-\lambda} & 0 \\ g & e_{\lambda} \end{bmatrix}, \quad \lambda > 0, \quad g \in L_\infty(\mathbb{R}), \tag{1.2}$$

where the notation e_δ ($\delta \in \mathbb{R}$) stands for the function $e_\delta(\xi) = e^{i\delta\xi}$ ($\xi \in \mathbb{R}$). It is known that the study of convolution equations on a finite interval of length λ is closely related to the study of Toeplitz operator with symbols of that form ([7]). Despite the apparent simplicity of its symbols, these operators present great difficulties and properties such as invertibility and Fredholmness have been studied only for particular classes of the non-diagonal element g. For instance, the case where g is an almost periodic function has been extensively studied in the literature (cf. [3, 4, 5, 7, 9, 13, 14, 15, 20, 21, 25, 26, 27, 28, 29]), although many questions remain unanswered.

This fourth chapter is divided in three sections. In the first two, we have g in (1.2) such that $0 \in \mathrm{sp}(g)$, while in the last one $0 \notin \mathrm{sp}(g)$. In fact, experience indicates that these two problems are of a different nature (see also Chapter 15 in [7]) and therefore it is natural to take different approaches to deal with them.

In Section 4.1, we are dealing with APP matrix symbols. A table method such as that presented in [13] and developed in [15] is used to obtain an almost periodic solution, with Fourier spectrum in a group $\alpha\mathbb{Z}+\beta\mathbb{Z}$ with given $\alpha, \beta \in \mathbb{R}$, to equation (1.1). In the first and third examples, the method presented in [25] can be also applied. In all these cases, applying mainly Theorems 3.4, 3.9, 3.11 and 3.17, we establish necessary and sufficient conditions for existence of a canonical factorization for G, thus for invertibility of T_G.

It should be noticed that, as regards Example 4.1.1, such conditions have been determined before in [3], where a wider class of matrix functions is studied. However, we deduce them here in a different and simpler way and, with this approach, those conditions also appear naturally in a simpler form. The results obtained in each example are compared with those obtained through a different approach by using the so-called Portuguese Transformation.

In Section 4.2 we are no longer restricted to almost periodic functions and we show how Theorem 3.13 can be used to study the invertibility of Toeplitz operators by reducing it to the study of other classes with simpler symbols.

In Section 4.3 we study a class of triangular matrix symbols G of the form (1.2) with a non-diagonal entry with a "gap around zero" of the form

$$g = a_- e^{-i\beta\xi} + a_+ e^{i\nu\xi} \,,$$

where $a_\pm \in H^\pm_\infty$ and $\nu, \beta \in]0, \lambda[$. In this case, we are able to determine a solution to the Riemann-Hilbert problem which is also valid for non APW functions g. Thus we start by studying the invertibility and, under some conditions, the Fredholmness of T_G (i.e., the existence of a Wiener-Hopf factorization of G). We consider the cases $\nu+\beta \geq \lambda$ and $\nu+\beta < \lambda$ separately. First we study the case where $\nu+\beta \geq \lambda$, thus generalizing some results of [12] regarding the canonical factorization of this class of symbols, and secondly we obtain conditions for existence of a Wiener-Hopf factorization of G when $\nu + \beta < \lambda$, which to our knowledge, has not been studied before (except for some AP polynomial functions g). Assuming some additional conditions on g, we proceed to study the existence of an APW factorization for G and, if it exists, we determine its AP partial indices.

Finally, in Chapter 5 we briefly explain the procedure known as the Portuguese Transformation. A detailed exposition of this method can be found in [7]; here it is summarized for the benefit of self-containedness.

It should be noticed that in all examples presented here we determine a solution to the Riemann-Hilbert problem (1.1), which can be seen as a first step towards the explicit construction of a factorization for G. However, the study of the solvability of a Riemann-Hilbert problem (1.1) has an interest that goes beyond that for at least two reasons: first, it can lead to conclusions on the existence and properties of a factorization of G but, if the latter does not exist, one still has some information on the solution to (1.1); second, from an explicit determination of a solution to (1.1) it may be possible to characterize new classes of matrix functions for which analogous conclusions can be obtained and thus widen the field of applications of the results.

Chapter 2

Preliminaries

This chapter provides the main background needed throughout this thesis.

Let $L_p(\mathbb{R})$, $1 < p < \infty$, be the Banach space of all complex valued functions f, (Lebesgue) measurable in $\mathbb{R}$, such that $|f|^p$ is integrable in $\mathbb{R}$, with the norm (cf. [16], [24])

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{\frac{1}{p}} .$$

Let $H_p^{\pm}$, $1 < p < \infty$, denote the Hardy spaces $H^p(\mathbb{C}^{\pm})$ (cf. [18]) and let us identify each function $\phi_{\pm} \in H_p^{\pm}$ with its boundary-value on $\mathbb{R}$, which is a function in $L_p(\mathbb{R})$ (cf. [24]). We have then that the space $L_p(\mathbb{R})$ admits a direct sum decomposition

$$L_p(\mathbb{R}) = H_p^+ \oplus H_p^-$$

and we denote by P^+ the projection of $L_p(\mathbb{R})$ onto H_p^+ parallel to H_p^- and by P^- its complementary projection, $P^- = I - P^+$ (cf. [16], [24]).

It is well known (cf. [24]) that $P^{\pm}$ can also be defined in terms of the *singular integral operator*

$$S_{\mathbb{R}} : L_p(\mathbb{R}) \to L_p(\mathbb{R}), \quad p > 1$$

$$S_{\mathbb{R}}f(t) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(\tau)}{\tau - t}\, d\tau, \qquad t \in \mathbb{R},$$

where the integral is understood in the sense of Cauchy's principal value, i.e.,

$$\frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(\tau)}{\tau - t}\, d\tau = \lim_{\delta \to 0^+} \frac{1}{\pi i}\int_{|\tau - t| > \delta} \frac{f(\tau)}{\tau - t}\, d\tau.$$

The operator $S_{\mathbb{R}}$ is bounded in $L_p(\mathbb{R})$ and $S_{\mathbb{R}}^2 = I$, where I is the identity operator in $L_p(\mathbb{R}), p > 1$ (cf. [24]). In fact, the two complementary projections satisfy (cf. [24])

$$P^+ = \tfrac{1}{2}(I + S_{\mathbb{R}}) \qquad \text{and} \qquad P^- = \tfrac{1}{2}(I - S_{\mathbb{R}}).$$

Let $L_\infty(\mathbb{R})$ be the Banach space of measurable functions which are essentially bounded in $\mathbb{R}$, with the norm (cf. [24])

$$\|f\|_\infty = \sup_{t \in \mathbb{R}} \mathrm{ess}\, |f(t)|.$$

We denote by $H_\infty^\pm$ the Hardy spaces $H^\infty(\mathbb{C}^\pm)$ (cf. [18]) and we identify each function $\phi_\pm \in H_\infty^\pm$ with its boundary-value on $\mathbb{R}$, which is a function in $L_\infty(\mathbb{R})$ (cf. [24]).

Let moreover e_μ, with $\mu \in \mathbb{R}$, be a function defined on $\mathbb{R}$ by

$$e_\mu(\xi) = e^{i\mu\xi}, \qquad \xi \in \mathbb{R}. \tag{2.1}$$

Definition 2.1. *([7]) The smallest closed subalgebra of $L_\infty(\mathbb{R})$ that contains all the functions e_μ ($\mu \in \mathbb{R}$) is denoted by AP and is called the algebra of almost periodic functions:*

$$AP := alg_{L_\infty(\mathbb{R})}\{e_\mu : \mu \in \mathbb{R}\}.$$

Let $AP^+ := alg_{L_\infty(\mathbb{R})}\{e_\mu : \mu \geq 0\}$ and $AP^- = alg_{L_\infty(\mathbb{R})}\{e_\mu : \mu \leq 0\}$. Clearly, $AP^\pm$ are closed subalgebras of $AP \cap H_\infty^\pm$ (cf. [7]).

A function $p : \mathbb{R} \to \mathbb{C}$ which can be represented as a finite sum of the form

$$p = \sum_{j=1}^{n} c_j e_{\mu_j},$$

where $n \in \mathbb{N}$, $c_j \in \mathbb{C}$ and $\mu_j \in \mathbb{R}$, is called an *almost periodic polynomial*. The collection of all almost periodic polynomials is denoted by APP. Obviously, AP is the closure of APP in $L_\infty(\mathbb{R})$ (cf. [7]).

Definition 2.2. *([7]) Let APW be the set of all functions $f \in AP$ which can be written in the form*

$$f = \sum_j c_j e_{\mu_j}, \qquad \textit{with} \qquad \sum_j |c_j| < \infty,$$

where $\mu_j \in \mathbb{R}$ and $c_j \neq 0$ for at most countably many j.

The coefficients c_j are called the *Bohr-Fourier* ([7]) or simply *Fourier coefficients* of f. The countable set of the real values of μ_j such that the corresponding Fourier coefficient c_j is different from zero is the *Fourier spectrum* of f and it is denoted by $\mathrm{sp}(f)$. Moreover, if $f = \sum_{j=1}^{n} c_j e_{\mu_j}$ with $n \in \mathbb{N}$, then $\mathrm{sp}(f) \subset \{\mu_1, \mu_2, \ldots, \mu_n\}$. It should be noticed that if $f \in APW$, then this notation is consistent, if $\mathrm{sp}(f)$ denotes the Fourier spectrum of f.

For $X = APW$ or APP, let X^+ and X^- be the set of all functions $f \in X$ for which $\mathrm{sp}(f) \subset [0, +\infty[$ and $\mathrm{sp}(f) \subset]-\infty, 0]$, respectively. Obviously, $APW^\pm$ are closed subalgebras of APW (cf. [7]).

If A is an algebra of complex valued functions, we denote by $\mathcal{G}(A)$ the group of invertible elements in the algebra A and by $A^{2\times 2}$ the algebra of 2×2 matrices with entries in A.

Now we introduce the notion of Wiener-Hopf factorization. For this we will need the following. Let

$$\lambda_\pm(\xi) = \xi \pm i \ , \qquad r(\xi) = (\lambda_- \lambda_+^{-1})(\xi) = \frac{\xi - i}{\xi + i} \ , \ \text{ for } \xi \in \mathbb{R}. \qquad (2.2)$$

Definition 2.3. *([16], [24]) Let $G \in \mathcal{G}(L_\infty(\mathbb{R}))^{2\times 2}$. G admits a Wiener-Hopf factorization relative to $L_p(\mathbb{R})$, $1 < p < \infty$, if G can be represented in the form*

$$G = G_- D G_+ \tag{2.3}$$

where D is a diagonal rational matrix of the form

$$D = diag\left(r^{k_1}, r^{k_2}\right) \ , \tag{2.4}$$

with $k_1, k_2 \in \mathbb{Z}$ and $k_1 \leq k_2$, and the factors $G_\pm$ are such that, for $q = \frac{p}{p-1}$,

$$\lambda_+^{-1} G_+ \in \left(H_q^+\right)^{2\times 2} \ , \quad \lambda_+^{-1} G_+^{-1} \in \left(H_p^+\right)^{2\times 2} \ , \tag{2.5}$$

$$\lambda_-^{-1} G_- \in \left(H_p^-\right)^{2\times 2} \ , \quad \lambda_-^{-1} G_-^{-1} \in \left(H_q^-\right)^{2\times 2} \ , \tag{2.6}$$

$G_- P^+ G_-^{-1} I$ *is an operator defined on a dense subset of $(L_p(\mathbb{R}))^2$*

and possessing a bounded extension to $(L_p(\mathbb{R}))^2$. (2.7)

We will usually omit referring to $L_p(\mathbb{R})$ and simply say that (2.3) is a Wiener-Hopf factorization.

The integers k_1, k_2 in (2.4) are called the *partial indices* of G and their sum is the total index of G, $\text{ind}(G) = k_1 + k_2$ (cf. [16], [24]).

The factorization (2.3) is said to be *canonical* if the partial indices k_1, k_2 are zero. If the factors in (2.3) are such that $G_+^{\pm 1} \in (H_\infty^+)^{2\times 2}$ and $G_-^{\pm 1} \in (H_\infty^-)^{2\times 2}$, then the factorization is bounded (cf. [16], [24]).

If G admits a bounded factorization, this factorization is a Wiener-Hopf factorization relative to any $L_p(\mathbb{R})$ (cf. [16], [24]).

Only for some class of matrix functions can we get this kind of factorization, for instance the class of Hölder continuous matrix functions $(C^\mu(\dot{\mathbb{R}}))^{2\times 2}$ for $0 < \mu < 1$, the Wiener algebra, $(W(\dot{\mathbb{R}}))^{2\times 2}$, or the class of APW matrix functions (cf. [7], [24]).

One of the sources of interest in the Wiener-Hopf factorization is the role that it plays in the theory of Toeplitz operators.

Let us establish some notation and definitions that will be needed in what follows.

Let X and Y be Banach spaces and $\mathcal{L}(X,Y)$ the linear space of all (bounded and linear) operators from X to Y. Let moreover $A \in \mathcal{L}(X,Y)$.

We define

$$\ker A = \{x \in X : Ax = 0\}, \qquad \operatorname{Im} A = \{Ax : x \in X\}$$

and

$$\operatorname{coker} A = Y \big/ \overline{\operatorname{Im} A}.$$

Definition 2.4. *([16], [17], [24]) A operator A is called Fredholm if and only if* $\dim\ker A$ *is finite,* $\dim\operatorname{coker} A$ *is finite and* $\operatorname{Im} A$ *is closed in Y. The integer*

$$\operatorname{Ind}(A) = \dim\ker A - \dim\operatorname{coker} A$$

is called the Fredholm index of A. The operator A is said to be semi-Fredholm if $\operatorname{Im} A$ *is closed in Y and* $\dim\ker A$ *or* $\dim\operatorname{coker} A$ *is finite.*

Taking this into account, for $G \in (L_\infty(\mathbb{R}))^{2\times 2}$, let us now state an important result which establish the relation between the study of invertibility and Fredholmness of a Toeplitz operator (cf. [8], [24])

$$T_G : (H_p^+)^2 \longrightarrow (H_p^+)^2, \;\; T_G\phi^+ = P^+(G\phi^+).$$

and the existence of a Wiener-Hopf factorization for the symbol of the Toeplitz operator, G.

Theorem 2.5. *([8], [16], [24]) A matrix function $G \in \mathcal{G}\,(L_\infty(\mathbb{R}))^{2\times 2}$ admits a Wiener-Hopf factorization in $L_p(\mathbb{R})$ if and only if T_G is a Fredholm operator on $L_p(\mathbb{R})$.*

If the operator T_G is Fredholm, the $\dim\ker T_G$ is the sum of the absolute values of the non-positive partial indices of G, while the dimension of coker T_G is given by the sum of the non-negative partial indices of G. In this case, the Fredholm index of T_G is symmetric to the total index of G, i.e., $\operatorname{Ind}(T_G) = -\operatorname{ind}(G)$ (cf. [7], [8], [24]).

Moreover, we also have:

Theorem 2.6. *([16]) A matrix function $G \in \mathcal{G}\left(L_\infty(\mathbb{R})\right)^{2\times 2}$ admits a canonical Wiener-Hopf factorization in $L_p(\mathbb{R})$ if and only if T_G is an invertible operator on $L_p(\mathbb{R})$.*

There are some classes of matrix functions G where Fredholmness is equivalent to invertibility of the Toeplitz operator T_G. This is the case, for instance, when $G \in (AP)^{2\times 2}$ and, in particular, when $G \in (APW)^{2\times 2}$ (cf. [7]). In fact, we have

Theorem 2.7. *([7]) If $G \in (AP)^{2\times 2}$, then T_G is Fredholm if and only if T_G is invertible.*

From this theorem we conclude that when G is an almost periodic matrix function, a Wiener-Hopf factorization if it exists then it must be canonical. However, if a Wiener-Hopf factorization for G does not exist, we can study the problem of AP factorization of G, which appears as a natural generalization of a Wiener-Hopf factorization for this kind of matrix functions.

Definition 2.8. *([7]) A matrix function $G \in \mathcal{G}(AP)^{2\times 2}$ is said to admit an AP factorization if it can be represented in the form*

$$G = G_- D G_+ \tag{2.8}$$

where

$$G_+^{\pm 1} \in (AP^+)^{2\times 2}, \qquad G_-^{\pm 1} \in (AP^-)^{2\times 2} \tag{2.9}$$

and

$$D = \mathrm{diag}(e_{\tau_1}, e_{\tau_2}) \,, \quad \tau_1, \tau_2 \in \mathbb{R}. \tag{2.10}$$

The numbers $\tau_1\, \tau_2$ are called the *AP partial indices* of G. An AP factorization with $D = I$ in (2.8), i.e., in which all AP partial indices are zero, is called a *canonical AP factorization* (cf. [7]).

If (2.8) holds but the factors $G_\pm$ satisfy stronger requirements $G_+^{\pm 1} \in (APW^+)^{2\times 2}$, $G_-^{\pm 1} \in (APW^-)^{2\times 2}$ (or $G_+^{\pm 1} \in (APP^+)^{2\times 2}$, $G_-^{\pm 1} \in (APP^-)^{2\times 2}$), we say that G admits an APW (resp. APP) *factorization* (cf. [7]).

To finish this chapter, it should be remarked that when $G \in (AP)^{2\times 2}$ admits a canonical AP factorization, then it obviously admits a canonical bounded Wiener-Hopf factorization. However, when $G \in (APW)^{2\times 2}$, we can say moreover that if a canonical Wiener-Hopf factorization of G exists then it is a canonical APW factorization.

Theorem 2.9. *([7]) If $G \in (APW)^{2\times 2}$, then the following propositions are equivalent:*

(i) T_G is Fredholm,

(ii) T_G is invertible,

(iii) G admits a canonical AP factorization,

(iv) G admits a canonical APW factorization.

Chapter 3

Corona conditions and factorization

This chapter is divided in three sections. While the first one is based on the results of [2] and [11], the other two sections are based on [9].

3.1 Corona problems and criteria for factorability

Many properties of Toeplitz operators with symbol G are connected with certain factorizations of their symbols. In fact, as was mentioned in Chapter 2, it is known that invertibility of Toeplitz operators with symbol $G \in (L_\infty(\mathbb{R}))^{2\times 2}$ is equivalent to existence of a canonical Wiener-Hopf factorization of G.

The idea of relating invertibility of such operators with the corona theorem appears in several works in this area (cf. for instance, [2], [4], [5], [9], [12] and [27]) but we will start by focusing our attention in [2]. In this paper, the question of invertibility of a Toeplitz operator with a matrix symbol

$G \in (L_\infty(\mathbb{R}))^{2\times 2}$ such that $\det G = 1$ is studied by analysing the solutions of an associated Riemann-Hilbert problem.

To state what we can call the main result of [2], we start by defining the following classes.

Definition 3.1. *We define*

$$CP^{\pm} = \{\phi_\pm = (\phi_{1\pm}, \phi_{2\pm}) \in \left(H_\infty^\pm\right)^2 : \inf_{\mathbb{C}^\pm} (|\phi_{1\pm}| + |\phi_{2\pm}|\,) > 0\} \tag{3.1}$$

and if $\phi_\pm \in CP^\pm$ we say that $(\phi_{1\pm}, \phi_{2\pm})$ is a corona pair in $\mathbb{C}^\pm$.

By the corona theorem (cf. [18], [22]), if $(\phi_{1+}, \phi_{2+}) \in CP^+$ then there exists a pair $(\tilde\phi_{1+}, \tilde\phi_{2+}) \in (H_\infty^+)^2$ such that

$$\phi_{1+}\tilde\phi_{1+} + \phi_{2+}\tilde\phi_{2+} = 1 \quad \text{in} \quad \mathbb{C}^+$$

(and analogously in $\mathbb{C}^-$, if $(\phi_{1-}, \phi_{2-}) \in CP^-$).

When we consider the question of existence and determination of a Wiener-Hopf factorization for a 2×2 matrix function G with entries in $L_\infty(\mathbb{R})$, we are naturally led to the study of Riemann-Hilbert problems of the form

$$G\phi_+ = \phi_-, \qquad \phi_\pm \in \left(H_\infty^\pm\right)^2. \tag{3.2}$$

In fact, is clear that, if G admits a bounded canonical factorization of the form (2.3), then the Riemann-Hilbert problem (3.2) admits two linearly independent solutions (ϕ_+, ϕ_-) and (ψ_+, ψ_-). In fact, it is enough to take ϕ_+, ψ_+ as the two columns in G_+^{-1} and ϕ_-, ψ_- as the two columns in G_-. If we assume moreover, as in [2], that $\det G = 1$, we can assume $\det G_+^{-1} = \det G_- = 1$, so that $\phi_\pm = (\phi_{1\pm}, \phi_{2\pm})$ and $\psi_\pm = (\psi_{1\pm}, \psi_{2\pm})$ are such that

$$\phi_{1\pm}\psi_{2\pm} - \phi_{2\pm}\psi_{1\pm} = 1. \tag{3.3}$$

Therefore, the corona problem in $\mathbb{C}^+$ with data (ϕ_{1+}, ϕ_{2+}) admits a solution given by ψ_+, and the corona problem in $\mathbb{C}^-$ with data (ϕ_{1-}, ϕ_{2-}) admits a solution given by ψ_-.

If G does not admit a bounded factorization, then the Riemann-Hilbert problem (3.2) may not admit any non-trivial (i.e., non-zero) solution. However, if this problem admits one solution, it is known that G has a canonical Wiener-Hopf factorization (relative to any $L_p(\mathbb{R})$) if ϕ_+ and ϕ_- satisfy certain conditions, as proved in [2]. We state that result as follows.

Theorem 3.2. *Let $G \in (L_\infty(\mathbb{R}))^{2\times 2}$ and $\det G = 1$. If there is a non-trivial solution to the Riemann-Hilbert problem (3.2) such that $\phi_\pm \in CP^\pm$, then G admits a canonical Wiener-Hopf factorization (relative to $L_p(\mathbb{R})$) and T_G is invertible.*

The factors in such a canonical factorization can be expressed in terms of the solutions to two associated corona problems, with data (ϕ_{1+}, ϕ_{2+}) (in $\mathbb{C}^+$) and (ϕ_{1-}, ϕ_{2-}) (in $\mathbb{C}^-$).

For $\varphi \in L_\infty(\mathbb{R})$ we define

$$\tilde{P}^\pm(\varphi) = \lambda_+ P^\pm \left(\lambda_+^{-1}\, \varphi\right) \tag{3.4}$$

where $\lambda_+(\xi) = \xi + i$ for $\xi \in \mathbb{R}$, as in (2.2). It is clear that $\lambda_+^{-1}\tilde{P}^\pm(\varphi) \in H_p^\pm$. Now we can state the following.

Theorem 3.3. *([2]) Let the assumptions of Theorem 3.2 be satisfied. Let $G = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be such that $\det G = 1$. Then a canonical Wiener-Hopf factorization for G (relative to $L_p(\mathbb{R})$) is $G = G_- G_+$ with*

$$G_- = \begin{bmatrix} \phi_{1-} & -\phi_{2-}^* + \phi_{1-}F_- \\ \phi_{2-} & \phi_{1-}^* + \phi_{2-}F_- \end{bmatrix}, G_+ = \begin{bmatrix} \phi_{1+}^* - \phi_{2+}F_+ & \phi_{2+}^* + \phi_{1+}F_+ \\ -\phi_{2+} & \phi_{1+} \end{bmatrix} \tag{3.5}$$

where $(\phi^*_{1\pm}, \phi^*_{2\pm})$ *is a solution to the corona problem with data* $(\phi_{1\pm}, \phi_{2\pm})$, *respectively, i.e.,* $\phi^*_{1\pm}, \phi^*_{2\pm} \in H^\pm_\infty$ *are such that ,*

$$\phi_{1\pm}\phi^*_{1\pm} + \phi_{2\pm}\phi^*_{2\pm} = 1 \ , \tag{3.6}$$

and

$$F_\pm = \tilde{P}^\pm(F), \tag{3.7}$$

where

$$F = -a\phi^*_{1-}\phi^*_{2+} + b\phi^*_{1-}\phi^*_{1+} - c\phi^*_{2-}\phi^*_{2+} + d\phi^*_{2-}\phi^*_{1+}. \tag{3.8}$$

In Theorem 3.2 we saw that the existence of a non-trivial solution to the Riemann-Hilbert problem such that $\phi_\pm \in CP^\pm$ is sufficient for existence of a canonical factorization of G, but in general it is not necessary. However, for some classes of matrix functions (such as the algebra of Hölder continuous functions, the Wiener algebra and $(APW)^{2\times 2}$) it is known that, if a Wiener-Hopf factorization exists, then it is bounded. We say in this case that $G \in \mathcal{B}$ and, for such matrix functions, the existence of a solution to (3.2) satisfying the corona conditions, i.e., such that $\phi_\pm \in CP^\pm$, is necessary and sufficient for existence of a (bounded) canonical factorization for G.

The following question can then be considered: having determined a non-trivial solution $\phi_\pm$ to the Riemann-Hilbert problem (3.2) such that we do not have $\phi_\pm \in CP^\pm$, would it be possible to find another solution $\psi_\pm$ to the same problem such that $\psi_\pm \in CP^\pm$?

The answer is negative, for the following reasons. If there was $\psi_\pm \in CP^\pm$ satisfying (3.2), then $G \in \mathcal{B}$ would admit a bounded canonical factorization $G = G_- G_+$ where we can assume that $\det G_\pm = 1$ and $\det G = 1$.

From (3.2), we have

$$G_+\phi_+ = G_-^{-1}\phi_- = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \in \mathbb{C}^2$$

so that

$$\phi_+ = G_+^{-1} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, \quad \phi_- = G_- \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

where $|k_1| + |k_2| > 0$. Let $k_1 \neq 0$ and let g_2^+, g_2^- be the second columns in G_+^{-1} and G_-, respectively; then

$$\det[\phi_+, g_2^+] = \det G_+^{-1} = 1.$$

Therefore, $\phi_+ \in CP^+$ and, analogously, we should have $\phi_- \in CP^-$, which contradicts our assumptions. Thus, we have just proved the following.

Theorem 3.4. *If $G \in \mathcal{B}$, G admits a canonical factorization if and only if we have $\phi_\pm \in CP^\pm$ for any solution to (3.2).*

For $G \in (APW)^{2\times 2} \subset \mathcal{B}$, the existence of a solution to the Riemann-Hilbert problem (3.2) satisfying the corona conditions is not only a necessary and sufficient condition for existence of a canonical bounded factorization for G, but also for the existence of a canonical APW factorization of G (cf. Theorem 2.9).

From the last theorems we see that a non-trivial solution to a Riemann-Hilbert problem of the form (3.2) can provide important information to the study of the Wiener-Hopf factorization of the matrix $G \in (L_\infty(\mathbb{R}))^{2\times 2}$. However, it seems natural to expect also important information from such a solution as regards AP factorization when it is not canonical. For APW symbols this is indeed true and we can also obtain results on the AP partial indices and factors, as shown by the following.

Theorem 3.5. *Let $G \in (APW)^{2\times 2}$ be such that $\det G = 1$. The Riemann-Hilbert problem (3.2) admits a solution $\phi_\pm \in (APW^\pm)^2$ such that*

$$\phi_- \in CP^- \tag{3.9}$$

$$\tilde{\phi}_+ = e_{-\delta}\phi_+ \in CP^+,\ \textit{for some}\ \delta \geq 0. \tag{3.10}$$

iff G *admits an* APW *factorization.*

In this case the AP *partial indices of* G *are* $\pm\delta$ *and we can take* $\phi_-, \tilde{\phi}_+$ *as the first columns in* G_- *and* G_+^{-1} *in (2.8), respectively.*

Proof. If G admits an APW factorization then it is clear that the Riemann-Hilbert problem (3.2) admits a solution satisfying (3.9) and (3.10), with δ being the non-negative AP partial index.

To prove the converse, it is enough to consider the case where $\delta > 0$, since for $\delta = 0$ the result is known (cf. Theorem 3.2). Since

$$\phi_- \;=\; (\phi_{1-}, \phi_{2-}) \in (APW^-)^2 \cap CP^- \tag{3.11}$$

$$\tilde{\phi}_+ \;=\; (\tilde{\phi}_{1+}, \tilde{\phi}_{2+}) \in (APW^+)^2 \cap CP^+, \tag{3.12}$$

there exist $\phi_-^* = (\phi_{1-}^*, \phi_{2-}^*) \in (APW^-)^2 \cap CP^-$ and $\phi_+^* = (\phi_{1+}^*, \phi_{2+}^*) \in (APW^+)^2 \cap CP^+$ such that

$$\phi_{1-}\phi_{1-}^* + \phi_{2-}\phi_{2-}^* = 1 \quad \text{ in } \mathbb{C}^-,$$

$$\tilde{\phi}_{1+}\phi_{1+}^* + \tilde{\phi}_{2+}\phi_{2+}^* = 1 \quad \text{ in } \mathbb{C}^+$$

([7], Theorem 12.1). If

$$G = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{3.13}$$

we can write, taking into account that $G(e_\delta \tilde{\phi}_+) = \phi_-$,

$$G \begin{bmatrix} \tilde{\phi}_{1+} & -\phi_{2+}^* \\ \tilde{\phi}_{2+} & \phi_{1+}^* \end{bmatrix} = \begin{bmatrix} \phi_{1-} & -\phi_{2-}^* \\ \phi_{2-} & \phi_{1-}^* \end{bmatrix} \begin{bmatrix} e_{-\delta} & F \\ 0 & e_\delta \end{bmatrix} \tag{3.14}$$

with F given by (3.8).

Since $F \in APW$, from (3.4) we have

$$\tilde{F}_\pm = e_{\pm\delta}\tilde{P}^\pm(F) \in APW^\pm \tag{3.15}$$

([2], Proposition 5.5). On the other hand,

$$\begin{bmatrix} e_{-\delta} & F \\ 0 & e_{\delta} \end{bmatrix} = \begin{bmatrix} 1 & \tilde{F}_{-} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_{-\delta} & 0 \\ 0 & e_{\delta} \end{bmatrix} \begin{bmatrix} 1 & \tilde{F}_{+} \\ 0 & 1 \end{bmatrix} \tag{3.16}$$

and it follows from (3.14), (3.15) and (3.16) that

$$G = G_{-} D G_{+}, \tag{3.17}$$

with

$$G_{-} = \begin{bmatrix} \phi_{1-} & -\phi_{2-}^{*} \\ \phi_{2-} & \phi_{1-}^{*} \end{bmatrix} \begin{bmatrix} 1 & \tilde{F}_{-} \\ 0 & 1 \end{bmatrix} \tag{3.18}$$

$$G_{+} = \begin{bmatrix} 1 & \tilde{F}_{+} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_{1+}^{*} & \phi_{2+}^{*} \\ -\tilde{\phi}_{2+} & \tilde{\phi}_{1+} \end{bmatrix} \tag{3.19}$$

$$D = \operatorname{diag}(e_{-\delta}, e_{\delta}) \tag{3.20}$$

is an APW factorization of G, with AP partial indices $\pm\delta$, such that $\phi_{-}, \tilde{\phi}_{+}$ are the first columns in G_{-} and G_{+}^{-1}, respectively. ■

Remark 3.6. *It is clear that the proof of the last theorem provides, through (3.18)-(3.20), formulas for the factors in an APW factorization of G.*

3.2 Corona conditions on a strip and the corona theorem

From the results of the previous section is clear the importance of knowing how to verify that a solution to a Riemann-Hilbert problem is a corona pair. Usually this is not an easy task, but may be simplified (as we will see below) by using an approach that takes advantage of some properties of the matrix G. So we start by establishing some notation regarding complex functions which are analytic in a strip.

Definition 3.7. *If*

$$S_{-\varepsilon_2,\varepsilon_1} = \{z \in \mathbb{C} : -\varepsilon_2 < \mathrm{Im}(z) < \varepsilon_1\}, \quad \textit{with} \;\; \varepsilon_1, \varepsilon_2 \in [0, +\infty], \tag{3.21}$$

we say that $H_\infty(S_{-\varepsilon_2,\varepsilon_1})$ is the space of functions that are analytic and bounded in $S_{-\varepsilon_2,\varepsilon_1}$.

Let $G \in (L_\infty(\mathbb{R}))^{2\times 2}$, with $\det G = \gamma$, satisfy

$$G \in (H_\infty(S_{-\varepsilon_2,\varepsilon_1}))^{2\times 2}, \qquad \gamma^{-1} \in H_\infty(S_{-\varepsilon_2,\varepsilon_1}). \tag{3.22}$$

If these conditions are satisfied and $\phi_\pm \in (H_\infty^\pm)^2$ is a non-trivial solution to the Riemann-Hilbert problem (3.2) then ϕ_+ admits an analytic and bounded extension to $\mathbb{C}^+ - i\varepsilon_2$ and ϕ_- admits an analytic and bounded extension to $\mathbb{C}^- + i\varepsilon_1$ (which we denote by $\phi_\pm$ as well, respectively).

Taking this into account, we can state the following.

Theorem 3.8. *Let*

$$G \in (H_\infty(S_{-\varepsilon_2,\varepsilon_1}))^{2\times 2}, \qquad \det G = \gamma \qquad \textit{and} \qquad \gamma^{-1} \in H_\infty(S_{-\varepsilon_2,\varepsilon_1}), \tag{3.23}$$

with $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$ and let $\phi_\pm = (\phi_{1\pm}, \phi_{2\pm}) \in (H_\infty^\pm)^2$ be such that $G\phi_+ = \phi_-$. Then (using the notation $S = S_{-\varepsilon_2,\varepsilon_1}$ for simplicity)

$$\inf_S(|\phi_{1+}| + |\phi_{1-}|) > 0 \implies \inf_S(|\phi_{1+}| + |\phi_{2+}|) > 0\,, \tag{3.24}$$

$$\inf_S(|\phi_{2+}| + |\phi_{2-}|) > 0 \implies \inf_S(|\phi_{1+}| + |\phi_{2+}|) > 0. \tag{3.25}$$

Proof. Assume that

$$G = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{3.26}$$

We prove only (3.24), since (3.25) can be obtained analogously. Let us assume that

$$\inf_S(|\phi_{1+}| + |\phi_{2+}|) = 0\,;$$

then there is a sequence $(\xi_n)_{n\in\mathbb{N}}$ of points in S such that

$$\phi_{1+}(\xi_n) \longrightarrow 0\,, \qquad \phi_{2+}(\xi_n) \longrightarrow 0.$$

Since, according to (3.2) and (3.26),

$$\phi_{1-} = a\phi_{1+} + b\phi_{2+}$$

and both a and b are analytic and bounded in S, we have also $\phi_{1-}(\xi_n) \longrightarrow 0$, which implies that

$$\inf_S(|\phi_{1+}| + |\phi_{1-}|\,) = 0.$$

■

The next theorem shows that several conditions that we might call of "corona type" are equivalent in a strip $S_{-\varepsilon_2,\varepsilon_1}$, with $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$, so that we are free to choose different pairs of functions in the set $\{\phi_{1+}, \phi_{1-}\,, \phi_{2+}, \phi_{2-}\}$ to express the same property.

Theorem 3.9. *Let $S = S_{-\varepsilon_2,\varepsilon_1}$ with $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$ and let G of the form (3.26) satisfy the conditions*

$$G \in (H_\infty(S))^{2\times 2}, \qquad \det G = \gamma \qquad \text{and} \qquad \gamma^{-1} \in H_\infty(S).$$

Let moreover $\phi_\pm = (\phi_{1\pm}, \phi_{2\pm}) \in (H^\pm_\infty)^2$ be such that $G\phi_+ = \phi_-$. Then the following propositions are equivalent:

(i) $\inf_S\,(|\phi_{1+}| + |\phi_{2+}|\,) > 0\,,$

(ii) $\inf_S\,(|\phi_{1-}| + |\phi_{2-}|\,) > 0\,,$

(iii) $\inf_S\,(|\phi_{2+}| + |\frac{1}{c}(\phi_{2-} - d\phi_{2+})|\,) > 0\,,$

(iv) $\inf_S\,(|\phi_{2-}| + |\frac{1}{c}(a\phi_{2-} - \gamma\phi_{2+})|\,) > 0.$

Proof. First we remark that, in (iii) and (iv), $\frac{1}{c}(\phi_{2-} - d\phi_{2+})$ and $\frac{1}{c}(a\phi_{2-} - \gamma\phi_{2+})$ are understood in the sense of analytic extensions, if necessary.

$(i) \Rightarrow (ii)$ In fact, if

$$\inf_S (|\phi_{1-}| + |\phi_{2-}|\ ) = 0$$

then, for some sequence (ξ_n) with $\xi_n \in S$ for all $n \in \mathbb{N}$, we have

$$\phi_{1-}(\xi_n) \longrightarrow 0\,, \quad \phi_{2-}(\xi_n) \longrightarrow 0.$$

Since

$$\begin{aligned} \phi_{1+} &= \gamma^{-1}(d\phi_{1-} - b\phi_{2-})\,, \\ \phi_{2+} &= \gamma^{-1}(-c\phi_{1-} + a\phi_{2-})\,, \end{aligned}$$

where γ^{-1}, a, b, c, d are bounded in S, we must have also

$$\phi_{1+}(\xi_n) \longrightarrow 0\,, \quad \phi_{2+}(\xi_n) \longrightarrow 0$$

so that

$$\inf_S (|\phi_{1+}| + |\phi_{2+}|\ ) = 0.$$

$(ii) \Rightarrow (i)$ analogously.

$(i) \Leftrightarrow (iii)$ because $\phi_{1+} = \frac{1}{c}(\phi_{2-} - d\phi_{2+})$, where the right-hand side of this equality is understood as the analytic extension at any point where c vanishes.

$(ii) \Leftrightarrow (iv)$ because $\phi_{1-} = \frac{1}{c}(a\phi_{2-} - \gamma\phi_{2+})$. ■

Definition 3.10. *Let* $S = S_{-\varepsilon_2,\varepsilon_1}$ *with* $\varepsilon_1, \varepsilon_2 \in [0, +\infty]$. *If two functions* $\psi_1, \psi_2 \in H_\infty(S)$ *satisfy*

$$\inf_S (|\psi_1| + |\psi_2|\ ) > 0,$$

then we say that $(\psi_1, \psi_2) \in CP(S)$.

As a consequence of Theorem 3.9, we have the following.

Theorem 3.11. *Let G satisfy condition (3.23) for some $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$ and let (ϕ_+, ϕ_-) be a solution to (3.2). Then $\phi_\pm \in CP^\pm$ iff*

$$\inf_{\mathbb{C}^+ + i\varepsilon_1} (|\phi_{1+}| + |\phi_{2+}|\,) > 0\,, \tag{3.27}$$

$$\inf_{\mathbb{C}^- - i\varepsilon_2} (|\phi_{1-}| + |\phi_{2-}|\,) > 0 \tag{3.28}$$

and one of the conditions $(i) - (iv)$ in Theorem 3.9 holds, for $S = S_{-\varepsilon_2, \varepsilon_1}$.

Proof. Let $\phi_\pm \in CP^\pm$. Since $\phi_+ \in CP^+$, we have $\phi_{1+}, \phi_{2+} \in CP(S \cap \mathbb{C}^+)$. By Theorem 3.9, we conclude that

$$\phi_{1-}, \phi_{2-} \in CP(S \cap \mathbb{C}^+). \tag{3.29}$$

Analogously, as $\phi_- \in CP^-$, we have

$$\phi_{1-}, \phi_{2-} \in CP(S \cap \mathbb{C}^-). \tag{3.30}$$

From (3.29) and (3.30) we conclude that

$$(\phi_{1-}, \phi_{2-}) \in CP(S).$$

So condition (ii) of Theorem 3.9 is satisfied and it is clear that (3.27) and (3.28) also hold.

Conversely, if one of the conditions $(i) - (iv)$ in Theorem 3.9 holds, then (i) and (ii) both hold which, together with (3.27) and (3.28), implies that $\phi_\pm \in CP^+$. ∎

Remark 3.12. *It is clear that, in order to apply these results, two preliminary questions have to be answered, namely how to determine a particular non-trivial solution to (3.2) and how to choose the best condition to verify, among $(i) - (iv)$ in Theorem 3.9.*

The answer to the second question must be given after evaluating which condition is easier to verify in each case and this obviously depends on the answer to the first question. As to the latter, it should be noticed that it has an interest of its own, independently from the fact that G admits any kind of factorization. The answer to that first question cannot be given in general, however we can characterize some classes of matrix functions G for which problem (3.2) can be solved, as we will see in Chapter 4.

These results can also be used to study the invertibility of some classes of Toeplitz operators by reducing it to the study of other classes with simpler symbols. Let us study this question in connection with the corona theorem, following Theorem 3.2 on the one hand and Theorem 3.11 on the other hand. We see that if G, ϕ_+ and ϕ_- satisfy the assumptions of Theorem 3.11 and moreover

$$\phi_+ \in CP(S_{-\varepsilon_2,+\infty}), \tag{3.31}$$

then $\phi_- \in CP^-$ iff (3.28) is satisfied. Analogously, if

$$\phi_- \in CP(S_{-\infty,\varepsilon_1}), \tag{3.32}$$

then $\phi_+ \in CP^+$ iff (3.27) is satisfied.

Thus we conclude the following.

Theorem 3.13. *Let G and $\tilde{G}$ be matrix functions in $(H_\infty(S_{-\varepsilon_2,\varepsilon_1}))^{2\times 2}$ with $\varepsilon_1, \varepsilon_2 \in [0,+\infty[$ and let $\gamma = \det G$, $\tilde{\gamma} = \det \tilde{G}$ be such that they admit a bounded canonical factorization and $\gamma^{-1}, \tilde{\gamma}^{-1} \in H_\infty(S_{-\varepsilon_2,\varepsilon_1})$. Let moreover $\tilde{G}$ admit a bounded canonical factorization.*

(i) If there are non-zero functions $\phi_+ \in (H_\infty^+)^2$, $\phi_-, \tilde{\phi}_- \in (H_\infty^-)^2$ such that

$$G\phi_+ = \phi_-, \quad \tilde{G}\phi_+ = \tilde{\phi}_-\,, \tag{3.33}$$

then G admits a canonical Wiener-Hopf factorization if condition (3.28) holds.

(ii) If there are non-zero functions $\phi_+,\ \tilde{\phi}_+ \in (H_\infty^+)^2,\ \phi_- \in (H_\infty^-)^2$ such that

$$G\phi_+ = \phi_-, \quad \tilde{G}\tilde{\phi}_+ = \phi_-\,, \tag{3.34}$$

then G admits a canonical Wiener-Hopf factorization if (3.27) holds.

Proof. If $\tilde{G}$ admits a bounded canonical Wiener-Hopf factorization and $\phi_+, \tilde{\phi}_-$ are non-zero solutions of the second equation in (3.33), then $\phi_+ \in CP^+, \tilde{\phi}_- \in CP^-$ and it follows from Theorems 3.11 and 3.9 that ϕ_+ satisfies condition (3.31). Thus $\phi_- \in CP^-$ if (3.28) holds and, from Theorem 3.2, we conclude that G admits a canonical Wiener-Hopf factorization.

The second part of the theorem can be proved analogously. ■

3.3 Corona conditions and common zeros

For several important classes of matrix symbols G, the behaviour of $\phi_\pm$ "at infinity", which is translated by conditions (3.27) and (3.28), is easy to study for big enough $\varepsilon_1, \varepsilon_2$. Therefore, verifying that $\phi_\pm \in CP^\pm$ (which implies the invertibility of the Toeplitz operator with symbol G when $\det G = 1$) is reduced to verifying one of the conditions $(i) - (iv)$ in Theorem 3.9. This, in turn, means roughly that we should be able to compare the zeros of two particular functions in the strip $S_{-\varepsilon_2,\varepsilon_1}$. It is clear, though, that verifying the corona conditions for a pair of functions in H_∞^+ (or H_∞^-) involves more than just studying their common zeros. Nonetheless in what follows we show that the verification of those conditions can be closely related to the study of the common points in two algebraic curves.

In what follows, S does not necessarily denote a strip as before; however we keep this notation having in mind its future applications.

Theorem 3.14. *Let $S \subset \mathbb{C}$ and let $\varphi : S \to \mathbb{C}^2$ be a function such that $\varphi(S)$ is bounded. Let $D \subset \mathbb{C}^2$ be such that $D \supset \overline{\varphi(S)}$ and let $F, H : D \to \mathbb{C}$ be continuous functions. Let moreover $f = F \circ \varphi$, $h = H \circ \varphi$. Then*

$$\inf_{\xi \in S} (\, |f(\xi)| + |h(\xi)| \,) = 0 \tag{3.35}$$

iff

$$\begin{cases} F(z, w) = 0 \\ H(z, w) = 0 \end{cases} \tag{3.36}$$

admits a solution $(z_0, w_0) \in \overline{\varphi(S)}$.

Proof. If (3.35) is satisfied, then there is a sequence (ξ_n), with terms $\xi_n \in S$ for all $n \in \mathbb{N}$, such that

$$f(\xi_n) \longrightarrow 0, \qquad h(\xi_n) \longrightarrow 0.$$

Therefore, defining $(z_n, w_n) = \varphi(\xi_n)$, we have

$$F(z_n, w_n) \longrightarrow 0, \qquad H(z_n, w_n) \longrightarrow 0. \tag{3.37}$$

Since (z_n, w_n) is bounded in $\mathbb{C}^2$, it admits a convergent subsequence, so that, without loss of generality, we can assume that (z_n, w_n) is convergent in $\mathbb{C}^2$.

Let

$$(z_0, w_0) = \lim_{n \to \infty} (z_n, w_n) \in \overline{\varphi(S)} \subset D.$$

Since F and H are continuous in D, we have, from (3.37),

$$F(z_0, w_0) = 0, \qquad H(z_0, w_0) = 0.$$

Thus (3.36) admits a solution $(z_0, w_0) \in \overline{\varphi(S)}$.

Conversely, if (3.36) is satisfied for some $(z_0, w_0) \in \overline{\varphi(S)}$, then

$$(z_0, w_0) = \lim_{n\to\infty} \varphi(\xi_n)$$

for some sequence (ξ_n) with terms in S.

Then

$$\lim_{n\to\infty} f(\xi_n) = \lim_{n\to\infty} F(\varphi(\xi_n)) = F(z_0, w_0) = 0$$

and, analogously,

$$\lim_{n\to\infty} h(\xi_n) = 0,$$

so that

$$\inf_{\xi\in S}(\,|f(\xi)| + |h(\xi)|\,) = 0.$$

■

Let us now consider a particular case, where $S = S_{-\varepsilon_2,\varepsilon_1}$ is defined as in (3.21), with $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$, and $\varphi : S \longrightarrow \mathbb{C}^2$ is given by

$$(z, w) = \varphi(\xi) = (e^{i\alpha\xi}, e^{-i\beta\xi}), \quad \xi \in S \tag{3.38}$$

for fixed $\alpha, \beta > 0$, $\frac{\alpha}{\beta} \notin \mathbb{Q}$. With these assumptions, let us establish conditions for a point $(z_0, w_0) \in \mathbb{C}^2$ to belong to $\overline{\varphi(S)}$. In order to do this, we first prove an auxiliary result.

Lemma 3.15. *Let $\alpha, \beta > 0$, $\frac{\alpha}{\beta} \in \mathbb{R}\backslash\mathbb{Q}$ and $\theta, \tilde{\theta} \in \mathbb{R}$. Then there exist sequences (m_k) and (n_k) of integer numbers such that*

$$\lim_{k\to\infty} u_k = 0,$$

where $u_k = \frac{1}{\alpha}\,(\theta + 2m_k\pi) + \frac{1}{\beta}\,(\tilde{\theta} + 2n_k\pi)$.

Proof. For every $k \in \mathbb{N}$, let $\varepsilon_k = \frac{\alpha}{2\pi k}$. From Kronecker's theorem in one dimension ([19]), there are $n_k \in \mathbb{N}$, $m_k \in \mathbb{Z}$ such that

$$\left| n_k\,\frac{\alpha}{\beta} + m_k + \frac{\alpha\tilde{\theta} + \beta\theta}{2\pi\beta} \right| < \varepsilon_k$$

which means that

$$\left|\frac{1}{\alpha}(\theta + 2m_k\pi) + \frac{1}{\beta}(\tilde{\theta} + 2n_k\pi)\right| < \frac{1}{k} \overset{k\to\infty}{\longrightarrow} 0.$$

■

With $S = S_{-\varepsilon_2,\varepsilon_1}$ and φ defined by (3.38), we have the following.

Theorem 3.16. *Let* (z_0, w_0) *be such that*

$$-\frac{1}{\alpha}\log|z_0| \in \,]-\varepsilon_2\,,\varepsilon_1\,[. \tag{3.39}$$

Then $(z_0, w_0) \in \overline{\varphi(S)}$ *iff*

$$\frac{1}{\alpha}\log|z_0| = -\frac{1}{\beta}\log|w_0|. \tag{3.40}$$

Proof. Let $(z_0, w_0) \in \overline{\varphi(S)}$; then there exists (ξ_n), with $\xi_n \in S$, such that

$$\begin{aligned}(z_0, w_0) &= \lim_{n\to\infty}\varphi(\xi_n)\\ &= \lim_{n\to\infty}(e^{i\alpha\xi_n}, e^{-i\beta\xi_n}).\end{aligned}$$

Let $z_n = e^{i\alpha\xi_n}$, $w_n = e^{-i\beta\xi_n}$. We have

$$\mathrm{i}(\xi_n) = -\frac{1}{\alpha}\log|z_n| = \frac{1}{\beta}\log|w_n|\,,$$

where $\mathrm{i}(\xi_n)$ means the imaginary part of ξ_n, and since $|z_n| \longrightarrow |z_0|$ and $|w_n| \longrightarrow |w_0|$, it follows that $\mathrm{i}(\xi_n)$ converges when $n \to \infty$ and

$$\frac{1}{\alpha}\log|z_0| = -\frac{1}{\beta}\log|w_0|.$$

Conversely, let $(z_0, w_0) \in \mathbb{C}^2$ be such that the previous equality holds. Let θ and $\tilde{\theta}$ be some fixed values of $\arg z_0$ and $\arg w_0$, respectively, and let

$$\begin{aligned}\xi_k &= -\frac{i}{\alpha}\log|z_0| + \frac{1}{\alpha}(\theta + 2m_k\pi),\\ \tilde{\xi}_k &= \frac{i}{\beta}\log|w_0| - \frac{1}{\beta}(\tilde{\theta} + 2n_k\pi),\end{aligned}$$

where (m_k) and (n_k) are sequences of integer numbers such that $\tilde{\xi}_k = \xi_k - u_k$ with $u_k \to 0$, according to Lemma 3.15. We have $\xi_k \in S$ and

$$\varphi(\xi_k) = (z_0, w_0\, e^{-i\beta u_k}),$$

so that

$$\lim_{k\to\infty} \varphi(\xi_k) = (z_0, w_0) \in \overline{\varphi(S)}.$$

■

As a consequence of Theorems 3.14 and 3.16, we can now establish necessary and sufficient conditions for some pairs of functions (f, h) to satisfy the corona conditions in a strip, meaning that f and h do not approach 0 simultaneously in the strip.

Theorem 3.17. *Let F, H be continuous functions (in $\mathbb{C}^2$) and let $S = S_{-\varepsilon_2,\varepsilon_1}$ with $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$ be such that, for all common zeros (z_0, w_0) of F and H, condition (3.39) is satisfied. Let moreover $f = F \circ \varphi$, $h = H \circ \varphi$ with $\varphi : S \longrightarrow \mathbb{C}^2$ defined by (3.38). Then*

$$\inf_{\xi\in S}(\,|f(\xi)| + |h(\xi)|\,) = 0$$

iff there is a solution (z_0, w_0) of (3.36) for which condition (3.40) is satisfied.

Proof. This is a direct consequence of Theorems 3.14 and 3.16. ■

Chapter 4

Applications to some classes of Toeplitz operators

In this chapter we apply the previous results to study the properties of Toeplitz operators T_G with triangular symbols of the form

$$G = \begin{bmatrix} e_{-\lambda} & 0 \\ g & e_{\lambda} \end{bmatrix}, \quad \lambda > 0\,, \quad g \in L_\infty(\mathbb{R})\,, \tag{4.1}$$

where for each $\delta \in \mathbb{R}$, e_δ denotes the function defined by (2.1), in relation with an appropriate factorization of G.

We approach this problem by studying the Riemann-Hilbert problem

$$G\phi_+ = \phi_-\,, \quad \phi_\pm \in \left(H_\infty^\pm\right)^2 . \tag{4.2}$$

In the examples that we consider here, although the solutions to (4.2) are explicitly given and can be checked directly, it seems useful to understand how they were obtained. Thus a brief explanation is given in every case.

Throughout we assume the notation $\sum_{j=i}^{n} x_j = 0$ if $i > n$.

4.1 Toeplitz operators with APP symbols

In this section we focus our attention on symbols G of the form (4.1), where g is an almost periodic polynomial. The study of these examples is based on the results of [9]. Necessary and sufficient conditions for invertibility of some Toeplitz operators with APP matrix symbol G will be established using the results of the previous chapter. In all the examples studied in this section we have $G \in \mathcal{B}$ (see Section 3.1), therefore the existence of a non-trivial solution to the Riemann-Hilbert problem (4.2) such that $\phi_{\pm} \in CP^{\pm}$ is necessary and sufficient for existence of a bounded canonical factorization of G and therefore for invertibility of T_G.

In each example, the results are compared with those obtained by using a procedure known as the Portuguese Transformation (see Chapter 5) which consists in reducing the factorization problem of one matrix to another one, which usually is simpler to study than the original one. Finally, we make some critical remarks about the application of the method explained in Chapter 3 and the Portuguese Transformation.

4.1.1 $g = ae_{\alpha} + b + ce_{-\beta}, \;\; 2\alpha + \beta = \lambda, \;\; \beta < \alpha$

Let g in (4.1) be a trinomial of the form

$$g = ae_{\alpha} + b + ce_{-\beta} \tag{4.3}$$

with

$$a, b, c \in \mathbb{C}\backslash\{0\}\,, \qquad \alpha, \beta \in]0,1[\,, \qquad \frac{\alpha}{\beta} \notin \mathbb{Q}\,, \tag{4.4}$$

$$2\alpha + \beta = \lambda, \qquad \beta < \alpha. \tag{4.5}$$

In this case, a non-trivial solution to the Riemann-Hilbert problem (4.2) is known from [15] and is given by

$$\phi_{1+} = 1 + \sum_{l=0}^{k}\Big(-\frac{a}{b}\Big)\Big(-\frac{c}{b}\Big)^{l} e_{\alpha-l\beta} - \sum_{l=-1}^{k-1}(k-l)\Big(-\frac{a}{b}\Big)^{2}\Big(-\frac{c}{b}\Big)^{l} e_{2\alpha-l\beta}\,, \tag{4.6}$$

$$\phi_{1-} = e_{-\lambda}\phi_{1+}\,, \tag{4.7}$$

$$\phi_{2+} = -\frac{a^2}{c}(k+1) + \frac{a^3}{b^2}\sum_{l=-1}^{k-1}(k-l)\left(-\frac{c}{b}\right)^{l} e_{\alpha-(l+1)\beta}\,, \tag{4.8}$$

$$\phi_{2-} = b + ce_{-\beta} - \frac{ac}{b}\left(-\frac{c}{b}\right)^{k} e_{\alpha-(k+1)\beta}\,, \tag{4.9}$$

with $k = \left[\frac{\alpha}{\beta}\right]$.

Knowing a solution to the Riemann-Hilbert problem, we now want to study this solution in order to establish necessary and sufficient conditions for existence of a (canonical) Wiener-Hopf factorization for the matrix G.

Taking into account the expressions of ϕ_{2+} and ϕ_{2-} given by (4.8) and (4.9), respectively, it is easy to see that $0 \in \mathrm{sp}(\phi_{2\pm})$, so that, for sufficiently big $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$, we have

$$\inf_{\mathbb{C}^+ + i\varepsilon_1} (\,|\phi_{1+}| + |\phi_{2+}|\,) \;>\; 0\,, \tag{4.10}$$

$$\inf_{\mathbb{C}^- - i\varepsilon_2} (\,|\phi_{1-}| + |\phi_{2-}|\,) \;>\; 0. \tag{4.11}$$

Therefore, from Theorem 3.11, to prove that $\phi_\pm \in CP^\pm$, it is enough to show that one of the conditions (i)-(iv) in Theorem 3.9 holds in the strip $S = S_{-\varepsilon_2,\varepsilon_1}$ for $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$ such that (4.10) and (4.11) hold. So we will show that

$$\inf_S \left(|\phi_{2-}| + \left| \frac{e_{-\lambda}\phi_{2-} - \phi_{2+}}{g} \right| \right) > 0. \tag{4.12}$$

It is clear that, for any strip S of the above-mentioned type, (4.12) is equivalent to

$$\inf_S \left(|\phi_{2-}| + \left| \frac{e_{-\beta}\phi_{2-} - e_{2\alpha}\phi_{2+}}{g} \right| \right) > 0.$$

With the notation

$$f = \phi_{2-}\,, \qquad h = \frac{e_{-\beta}\phi_{2-} - e_{2\alpha}\phi_{2+}}{g}$$

we have

$$f = F \circ \varphi \qquad \text{and} \qquad h = H \circ \varphi\,,$$

with $\varphi : S \longrightarrow \mathbb{C}^2$ defined by (3.38) and

$$\begin{aligned} F(z,w) &= b + cw - \frac{ac}{b}\left(-\frac{c}{b}\right)^k zw^{k+1}\,, \\ H(z,w) &= \frac{N(z,w)}{D(z,w)}\,, \end{aligned}$$

with

$$N(z,w) = wF(z,w) + \left(\frac{a^2}{c}(k+1) - \frac{a^3}{b^2}\sum_{l=-1}^{k-1}(k-l)\left(-\frac{c}{b}\right)^l zw^{l+1}\right) z^2$$

and

$$D(z,w) = az + b + cw.$$

Notice that H is a polynomial given by

$$H(z,w) = \left(1 + \sum_{l=0}^{k}\left(-\frac{a}{b}\right)\left(-\frac{c}{b}\right)^l zw^l - \sum_{l=-1}^{k-1}(k-l)\left(-\frac{a}{b}\right)^2\left(-\frac{c}{b}\right)^l z^2w^l\right) w\,,$$

as well as F, so that F and H are continuous in $\mathbb{C}^2$.

Following Theorem 3.17, we now study the solutions of

$$\begin{cases} F(z,w) = 0 \\ H(z,w) = 0. \end{cases} \tag{4.13}$$

For any solution of (4.13), we have $w \neq 0$ and $z \neq 0$ and from the first equation of (4.13) we get

$$z = (b + cw)\frac{b}{ac}\left(-\frac{b}{c}\right)^k w^{-(k+1)}. \tag{4.14}$$

On the other hand, it is clear that all the solutions to (4.13) must also satisfy the equalities

$$\begin{cases} F(z,w) = 0 \\ \\ N(z,w) = 0 \end{cases} \quad \text{with} \quad z \neq 0, \quad w \neq 0. \tag{4.15}$$

Therefore, we start by determining all the solutions to (4.15), which, taking (4.14) into account, is equivalent to

$$\begin{cases} z = (b+cw)\frac{b}{ac}\left(-\frac{b}{c}\right)^k w^{-(k+1)} \\ \\ (k+1)w^{k+1} - \left(1+\frac{c}{b}w\right)\sum_{j=1}^{k+1} j\left(-\frac{b}{c}\right)^j w^{-j+k+1} = 0 \end{cases}$$

(with $z \neq 0,\ w \neq 0$).

This system of equations admits $k+1$ solutions which are

$$(z_0, w_0) = \left(-\frac{b}{a}\,\frac{(k+2)^k}{(k+1)^{k+1}}, -\frac{b}{c}\,\frac{k+1}{k+2}\right) \tag{4.16}$$

and

$$(z_j, w_j) = \left(-\frac{b}{a}\,(1-\alpha_j^{-1}), -\frac{b}{c}\,\alpha_j^{-1}\right)$$

where $\alpha_j = e^{i\frac{j2\pi}{k+1}}$, $j = 1, 2, \ldots, k$.

It is clear that $H(z_0, w_0) = 0$ (since $N(z_0, w_0) = 0$, $D(z_0, w_0) \neq 0$). As to the points (z_j, w_j), $j = 1, \ldots, k$, it is easy to see that $H(z_j, w_j) \neq 0$.

So we conclude that the unique solution to the system (4.13) is (z_0, w_0) given by (4.16).

Let $S = S_{-\varepsilon_2, \varepsilon_1}$ be such that (3.39) is satisfied, as well as (4.10) and (4.11). From Theorem 3.17 we conclude that

$$\inf_S \left(|\phi_{2-}| + \left| \frac{e_{-\lambda}\phi_{2-} - \phi_{2+}}{g} \right| \right) > 0$$

iff

$$\left|\frac{a}{b}\right|^{\frac{\beta}{\alpha}} \left|\frac{c}{b}\right| \neq \frac{k+1}{k+2} \left| \frac{(k+2)^k}{(k+1)^{k+1}} \right|^{\frac{\beta}{\alpha}}, \tag{4.17}$$

As in this case $G \in \mathcal{B}$, from Theorems 3.2 and 3.4 we conclude that T_G is invertible (and G admits a canonical bounded factorization) iff (4.17) holds, which, although in a different form, is equivalent to the condition obtained in [3].

This example can also be studied by the Portuguese Transformation (cf. Chapter 5). By using this procedure twice, the matrix (4.1) satisfying (4.3)-(4.5) is reduced to a case studied in Theorem 6.1 in [5] and the invertibility condition for T_G given by (4.17) can be confirmed. Since this example was previously studied in [3], we will not explain here how the Portuguese Transformation could be applied in this case.

4.1.2 $g = ae_\alpha + b + c_1 e_{-\beta} + c_2(k-1)e_{-2\beta}$, $2\alpha + \beta = \lambda$, $\beta > \alpha$, $3\alpha - 2\beta < 0$

For a trinomial g in (4.1) of the form $g = ae_\alpha + b + ce_{-\beta}$, necessary and sufficient conditions for existence of a Wiener-Hopf factorization for the matrix G, when $2\alpha + \beta = \lambda$ and $\frac{\alpha}{\beta} \notin \mathbb{Q}$, was given in [3] (whether $\beta > \alpha$ or $\beta < \alpha$) and, in particular, it coincides with (4.17), when $\beta < \alpha$.

However, when we look for explicit solutions to the Riemann-Hilbert problem (4.2), we see that the case where $\beta > \alpha$ admits simpler almost periodic polynomial solutions (cf. [15], [25]). Therefore, taking the approach of the present paper, it is simpler to verify if a non-trivial solution to (4.2) satisfies the corona conditions when $\beta > \alpha$. This is the main reason why we preferred to study the (more difficult) case where $\beta < \alpha$ in the previous example.

Now, it seems natural to proceed to a next step in this study by adding more points to the spectrum of g, thus obtaining conditions for existence of a Wiener-Hopf factorization of matrix functions in a wider class.

Therefore, we take now g, not as a trinomial , but as an almost-periodic polynomial admitting more than three points in its spectrum. In fact, if

$$g = ae_\alpha + b + \sum_{j=1}^{k} c_j\, e_{-j\beta}\,, \qquad k = \left[\frac{\lambda}{\beta}\right], \tag{4.18}$$

with

$$a, b \in \mathbb{C}\backslash\{0\}\,, \qquad \sum_{j=1}^{k} |c_j| \neq 0, \qquad \alpha, \beta \in]0, 1[\,, \qquad \frac{\alpha}{\beta} \notin \mathbb{Q}\,, \tag{4.19}$$

$$n\alpha + \beta = \lambda, \qquad \beta > (n-1)\alpha\,, \qquad n \in \mathbb{N}\,, \tag{4.20}$$

a non-trivial almost periodic polynomial solution always exists and can be determined using a table method of the type presented in [13] and [15].

We consider here only the case where $n = 2$ and some additional conditions are satisfied. A similar study can be carried out in the general case (4.18) although at the cost of greater computational difficulties.

So let g in (4.1) be of the form (4.18) with $n = 2$ (which implies that $k = 1$ or $k = 2$),

$$g = ae_{\alpha} + b + c_1 e_{-\beta} + c_2(k-1)e_{-2\beta} \tag{4.21}$$

where

$$2\alpha + \beta = \lambda, \qquad \beta > \alpha, \qquad 3\alpha - 2\beta < 0. \tag{4.22}$$

If $c_2(k-1) = 0$ in (4.21), we have a trinomial and it was proved in [25] that, in this case, there is an almost periodic polynomial solution to (4.2) if and only if $\alpha + \beta > \frac{\lambda}{2}$. However, since g in (4.21) is a quadrinomial, from this result we cannot conclude about the existence of an APP solution to the Riemann-Hilbert problem. But we will show that there is indeed such a solution and study it to obtain conditions for existence of a generalized factorization of G.

In fact, a first solution to the Riemann-Hilbert problem (4.2) can be obtained as we said before. In the following table, the Fourier spectrum of ϕ_{1+} (with points of the form $j\alpha - l\beta$) is represented for the case where $k = 2$ and $c_1\,, c_2 \neq 0$. In all other cases, $\mathrm{sp}(\phi_{1+})$ is contained in subset of $\alpha\mathbb{Z} + \beta\mathbb{Z}$ and can be represented in a similar way.

$j\backslash l$	-1	0	1
0		0	
1		α	
2	$2\alpha+\beta$	2α	$2\alpha-\beta$
3		3α	

Table 4.1.2.1: Fourier spectrum of ϕ_{1+}

The Fourier coefficients of ϕ_{1+} can be easily obtained from this table (using the same reasoning as in [13] and [15]). A non-trivial solution to the Riemann-Hilbert problem (4.2) is thus determined and we get, with $\tilde{c_2} = c_2(k-1)$,

$$\begin{aligned}\phi_{1+} &= (2bc_1^2 - b^2\tilde{c_2}) + (ab\tilde{c_2} - 2ac_1^2)e_\alpha - a^2\tilde{c_2}e_{2\alpha} + \frac{a^3\tilde{c_2}}{b}e_{3\alpha}\\ &\quad + 2a^2c_1e_\lambda - \frac{a^2c_1\tilde{c_2}}{b}e_{2\alpha-\beta}\,, \end{aligned} \tag{4.23}$$

$$\begin{aligned}\phi_{1-} &= 2a^2c_1 + (ab\tilde{c_2} - 2ac_1^2)e_{-\alpha-\beta} - a^2\tilde{c_2}e_{-\beta} + \frac{a^3\tilde{c_2}}{b}e_{\alpha-\beta}\\ &\quad + (2bc_1^2 - b^2\tilde{c_2})e_{-\lambda} - \frac{a^2c_1\tilde{c_2}}{b}e_{-2\beta}\,, \end{aligned} \tag{4.24}$$

$$\phi_{2+} = -2a^2c_1b - 2a^3c_1e_\alpha - \frac{a^4\tilde{c_2}}{b}e_{2\alpha-\beta}\,, \tag{4.25}$$

$$\begin{aligned}\phi_{2-} &= (2b^2c_1^2 - b^3\tilde{c_2}) + (2bc_1^3 - b^2c_1\tilde{c_2})e_{-\beta} + (abc_1\tilde{c_2} - 2ac_1^3)e_{\alpha-\beta}\\ &\quad + (-a^2\tilde{c_2}^2 - \frac{a^2c_1^2\tilde{c_2}}{b})e_{2\alpha-2\beta} + (2bc_1^2\tilde{c_2} - b^2\tilde{c_2}^2)e_{-2\beta}\\ &\quad + (ab\tilde{c_2}^2 - 2ac_1^2\tilde{c_2})e_{\alpha-2\beta} - \frac{a^2c_1\tilde{c_2}^2}{b}e_{2\alpha-3\beta} + \frac{a^3\tilde{c_2}^2}{b}e_{3\alpha-2\beta}\,. \end{aligned} \tag{4.26}$$

Knowing a non-trivial solution to the Riemann-Hilbert problem, we now want to establish necessary and sufficient conditions for existence of a Wiener-Hopf factorization for the matrix G.

It is easy to see that, for sufficiently big $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$, we have

$$\inf_{\mathbb{C}^+ + i\varepsilon_1} (\,|\phi_{1+}| + |\phi_{2+}|\,) \;>\; 0\,, \tag{4.27}$$

$$\inf_{\mathbb{C}^- - i\varepsilon_2} (\,|\phi_{1-}| + |\phi_{2-}|\,) \;>\; 0, \tag{4.28}$$

because, taking into account that $|c_1| + |c_2| \neq 0$, if $c_1 = 0$, then ϕ_{1+} and ϕ_{2-} have non-trivial constant terms and, if $c_1 \neq 0$, then ϕ_{1-} and ϕ_{2+} have non-trivial constant terms.

So, from Theorem 3.11, to prove that $\phi_\pm \in CP^\pm$ it is enough to show that one of the conditions (i)-(iv) in Theorem 3.9 holds in a strip $S = S_{-\varepsilon_2,\varepsilon_1}$ for $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$ such that (4.27) and (4.28) hold. So we will show that

$$\inf_S \left(|\phi_{1+}| + |\phi_{2+}| \right) > 0. \tag{4.29}$$

It is clear that, for a strip S of the above-mentioned type, (4.29) is equivalent to

$$\inf_S \left(|e_{-\beta}\phi_{1+}| + |\phi_{2+}| \right) > 0.$$

With the notation

$$f = e_{-\beta}\phi_{1+}\,, \qquad\qquad h = \phi_{2+}$$

we have

$$f = F \circ \varphi \qquad \text{and} \qquad h = H \circ \varphi\,, \tag{4.30}$$

with $\varphi : S \longrightarrow \mathbb{C}^2$ defined by (3.38) and

$$\begin{aligned} F(z,w) &= (2bc_1^2 - b^2\tilde{c}_2)w + (ab\tilde{c}_2 - 2ac_1^2)zw - a^2\tilde{c}_2 z^2 w + \frac{a^3\tilde{c}_2}{b} z^3 w \\ &\quad + 2a^2c_1z^2 - \frac{a^2c_1\tilde{c}_2}{b} z^2w^2\,, \\ H(z,w) &= -2a^2c_1b - 2a^3c_1z - \frac{a^4\tilde{c}_2}{b} z^2 w\,. \end{aligned}$$

Let us assume firstly that $c_1 \neq 0$.

The functions F and H are continuous in $\mathbb{C}^2$, so, following Theorem 3.17, we now study the solutions of

$$\begin{cases} F(z,w) = 0 \\ H(z,w) = 0. \end{cases} \tag{4.31}$$

For any solution of (4.31), we have $w \neq 0$ and $z \neq 0$ because $c_1 \neq 0$. The second equation of (4.31) is equivalent to

$$a^2 \tilde{c_2} z^2 w = -2bc_1(az+b). \tag{4.32}$$

We remark that, if $b\tilde{c_2} = 4c_1^2$, then there is no solution to the system (4.31). So, in this case, we conclude that the matrix G admits a canonical factorization.

Let us consider now $b\tilde{c_2} \neq 4c_1^2$. Replacing (4.32) in the first equation of (4.31), we obtain

$$z_0 = \frac{b}{a}\left(\frac{b\tilde{c_2} - 4c_1^2}{4c_1^2}\right). \tag{4.33}$$

Replacing (4.33) in (4.32), we determine the solution to the system (4.31), which is given by

$$\begin{cases} z_0 = \frac{b}{a}\left(\frac{b\tilde{c_2}-4c_1^2}{4c_1^2}\right) \\ w_0 = -\frac{8bc_1^3}{(b\tilde{c_2}-4c_1^2)^2}. \end{cases}$$

Let $S = S_{-\varepsilon_2,\varepsilon_1}$ be such that (3.39) is satisfied, as well as (4.27) and (4.28). From Theorem 3.17 we conclude that

$$\inf_S \left(|\phi_{1+}| + |\phi_{2+}| \right) > 0$$

iff

$$|a|^{\frac{\beta}{\alpha}} |2c_1|^{2\frac{\beta}{\alpha}-3} \neq |b|^{\frac{\beta}{\alpha}+1} |b\tilde{c_2} - 4c_1^2|^{\frac{\beta}{\alpha}-2}, \tag{4.34}$$

Therefore from Theorems 3.2 and 3.4 we conclude that T_G is invertible (and G admits a canonical bounded factorization) iff (4.34) holds.

If $c_1 = 0$ and $k = 2$, then we have a trinomial $g = aE^{\alpha} + b + c_2 E^{-2\beta}$ with $2\beta + \alpha > \lambda$ and G admits a canonical factorization (cf. [13]).

We have thus proved the following theorem.

Theorem 4.1. *([9]) Let G be given by (4.1), where g takes the form (4.21), with α, β satisfying (4.19) and (4.22). Then a solution to the Riemann-Hilbert problem (4.2) is given by (4.23)-(4.26) and G admits a canonical bounded factorization iff*

$$|a|^{\frac{\beta}{\alpha}}|2c_1|^{2\frac{\beta}{\alpha}-3} \neq |b|^{\frac{\beta}{\alpha}+1}|b\tilde{c_2} - 4c_1^2|^{\frac{\beta}{\alpha}-2}, \tag{4.35}$$

where we replace the right-hand side by 0 if $b\tilde{c_2} - 4c_1^2 = 0$ and $\frac{\beta}{\alpha} - 2 < 0$.

Remark 4.2. *If we take $c_2 = 0$ in (4.21), we obtain from (4.35) the condition*

$$\left|\frac{a}{b}\right|^{\frac{\beta}{\alpha}} \left|\frac{c_1}{b}\right| \neq \frac{1}{2},$$

which is the same condition that was already obtained in [3] for the trinomial case.

We will now apply the Portuguese Transformation in this case, leading to the same conclusions that we have already obtained.

In fact, applying twice this algorithm, the study of the matrix (4.1), with g satisfying (4.19), (4.21) and (4.22), is reduced to an already studied case. Now we will give a brief description of the procedure in this case.

As explained in Chapter 5, for

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

the matrices G and JG^*J are factorable only simultaneously. So now let us consider instead of G, the matrix

$$JG^*J = \begin{bmatrix} e_{-\lambda} & 0 \\ \overline{g} & e_{\lambda} \end{bmatrix} \tag{4.36}$$

where $\overline{g} = \overline{a}e_{-\alpha} + \overline{b} + \overline{c}_1 e_{\beta} + \overline{\tilde{c}}_2 e_{2\beta}$, $\quad \overline{\tilde{c}}_2 = \overline{c}_2(k-1)$ for $k = 1$ or $k = 2$.

As $-\alpha \in \mathrm{sp}(\overline{g})$ we can write

$$e_\alpha \overline{g} = \overline{a} + \overline{b} e_\alpha + \overline{c}_1 e_{\alpha+\beta} + \overline{\overline{c}}_2 e_{\alpha+2\beta} \tag{4.37}$$

where

$$q_0 = \overline{a}\,, \quad q_1 = \overline{b}\,, \quad q_2 = \overline{c}_1\,, \quad q_3 = \overline{\overline{c}}_2 \tag{4.38}$$

and

$$\gamma_1 = \alpha\,, \quad \gamma_2 = \alpha+\beta\,, \quad ,\gamma_3 = \alpha + 2\beta. \tag{4.39}$$

By the Portuguese Transformation, the matrix (4.36) is AP factorable only simultaneously with

$$\begin{bmatrix} e_{-\alpha} & 0 \\ g_1 & e_\alpha \end{bmatrix}, \quad \text{where} \quad \alpha < \lambda\,. \tag{4.40}$$

Here g_1 is an AP polynomial explicitly constructed from $\overline{g}$ (cf. Chapter 5)

$$\begin{aligned} g_1 &= f_2 e_{-\lambda} \\ &= \sum_{(n,\gamma)-\lambda<\alpha} q_0^{-1} d_n e_{(n,\gamma)-\lambda} \end{aligned}$$

where d_n is defined by (5.16), $(n,\gamma) := n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$ and $\gamma_1, \gamma_2, \gamma_3$ are given by (4.39).

By this construction, $\mathrm{sp}(g_1)$ lies in the set

$$\mathrm{sp}(g_1) \subset \{(n_1+n_2+n_3)\alpha + (n_2+2n_3)\beta - \lambda \;:\; n_1, n_2, n_3 \in \mathbb{Z}^+\} \cap (-\alpha, \alpha). \tag{4.41}$$

It is not difficult to see that all the terms of $\mathrm{sp}(g_1)$ in the interval $(-\alpha,\alpha)$ are of the form (4.41) with: $n_1 = 3\,,\, n_2 = n_3 = 0$ or $n_1 = 4\,,\, n_2 = n_3 = 0$ or $n_1 = n_2 = 1\,,\, n_3 = 0$ or $n_1 = n_2 = 0\,,\, n_3 = 1$.

Consequently,

$$\mathrm{sp}(g_1) \subset \{\, -\beta + \alpha\,,\ 0\,,\ 2\alpha - \beta\,,\ \beta - \alpha\,\}.$$

The coefficients of g_1 can be computed using (5.16) (cf. Chapter 5), taking into account the particular values of n_1, n_2, n_3 and (4.37)-(4.39).

So in this case g_1 is given by

$$g_1 = -\frac{\overline{b}^{\,3}}{\overline{a}^{\,4}} e_{-\beta+\alpha} + \frac{2\overline{b}\overline{c}_1}{\overline{a}^{\,3}} + \frac{\overline{b}^{\,4}}{\overline{a}^{\,5}} e_{2\alpha-\beta} - \frac{\overline{\overline{c}}_2}{\overline{a}^{\,2}} e_{\beta-\alpha}. \tag{4.42}$$

Since the matrix (4.40) with g_1 given by (4.42) was not studied before, we have to apply one more time the Portuguese Transformation, but now to the matrix function (4.40). Analogously, we see that this matrix is factorable iff the matrix

$$\begin{bmatrix} e_{-\beta+\alpha} & 0 \\ g_2 & e_{\beta-\alpha} \end{bmatrix} \quad \text{where} \quad \beta - \alpha < \alpha \tag{4.43}$$

is also factorable with

$$\operatorname{sp}(g_2) \subset \{ -2\alpha + \beta \,,\ 0 \,,\ 2\beta - 3\alpha \}.$$

The respective coefficients can be computed by formula (5.16) and we obtain:

$$g_2 = -2\frac{\overline{a}^{\,5}\overline{c}_1}{\overline{b}^{\,5}} e_{-2\alpha+\beta} - \frac{\overline{a}^{\,3}}{\overline{b}^{\,2}} - \frac{\overline{a}^{\,6}}{\overline{b}^{\,7}} (4\overline{c}_1^{\,2} - b\overline{\overline{c}}_2) e_{2\beta-3\alpha}.$$

If $\overline{c}_1 = 0$, then $\operatorname{sp}(g_2)$ lies to the right of 0. This situation falls into the so called one sided case (see [7], Theorem 14.3), in which the matrix (4.43) admits a canonical factorization. Consequently, the matrix (4.36) admits also a canonical factorization. In a similar way, the matrices (4.43) and (4.36) admit a canonical factorization when $4\overline{c}_1^{\,2} = b\overline{\overline{c}}_2$ holds (see [7], Theorem 14.4).

If $\overline{c}_1 \neq 0$ and $4\overline{c}_1^{\,2} \neq b\overline{\overline{c}}_2$, the matrix (4.43) satisfies conditions of Theorem 6.1 in [5] with an obvious change of notation. So we conclude that the matrix (4.43) (and therefore the matrix (4.36)) admits a canonical factorization iff

$$\left| \frac{\overline{a}^{\,6}}{\overline{b}^{\,7}} (4\overline{c}_1^{\,2} - b\overline{\overline{c}}_2) \right|^{\frac{2\alpha-\beta}{2\beta-3\alpha}} \left| 2\frac{\overline{a}^{\,5}\,\overline{c}_1}{\overline{b}^{\,5}} \right| \neq \left| \frac{\overline{a}^{\,3}}{\overline{b}^{\,2}} \right|^{\frac{2\alpha-\beta}{2\beta-3\alpha}+1},$$

which, after some computation, we see that it is equivalent to condition (4.35).

4.1.3 $g = ae_\alpha + b + ce_{-\beta},\ \alpha + \beta < \lambda,\ n\alpha - \beta = \lambda,\ n \geqslant 2$

Let us now consider a matrix function G of the form (4.1), where g is given by

$$g = ae_\alpha + b + ce_{-\beta} \tag{4.44}$$

with

$$a, b, c \in \mathbb{C}\backslash\{0\}\,, \qquad \alpha, \beta \in]0,1[\,, \qquad \frac{\alpha}{\beta} \notin \mathbb{Q}\,, \tag{4.45}$$

$$\alpha + \beta < \lambda \qquad \text{and} \qquad n\alpha - \beta = \lambda, \qquad \text{for} \quad n \geqslant 2. \tag{4.46}$$

Let us assume moreover that

$$\alpha + \beta > \max\{\lambda - \alpha, \lambda - \beta\}. \tag{4.47}$$

Taking into account (4.46), we see that (4.47) is equivalent to

$$\begin{cases} \frac{3\lambda}{2n+1} < \alpha < \frac{2\lambda}{n+1}\,, & \text{if} \quad 2 \leqslant n < 4; \\ \\ \frac{2\lambda}{n+2} < \alpha < \frac{2\lambda}{n+1}\,, & \text{if} \quad\ n \geqslant 4. \end{cases}$$

With these conditions, it is easy to see that $\alpha + \beta > \frac{\lambda}{2}$ and we can conclude that (4.2) admits an almost periodic polynomial solution and determine it according to [25]. Nevertheless, to determine this solution, it is also possible (and it may be advantageous in this case) to use a table method such as that used in the previous example, since it gives what might be called a "graphical" insight of the problem, namely as to understanding the differences between the even and odd cases. In the following tables, the Fourier spectrum of ϕ_{1+} (with points of the form $j\alpha - l\beta$) is represented, considering the cases where n is even or odd separately.

$j \backslash l$	0	1	2
0	0		
1	α		
2	2α		
$\vdots$	$\vdots$		
$\frac{n}{2}$	$\frac{n}{2}\,\alpha$	$\frac{n}{2}\,\alpha - \beta$	
$\vdots$		$\vdots$	
$n-1$		$(n-1)\alpha - \beta$	$(n-1)\alpha - 2\beta$
n		$n\alpha - \beta$	

Table 4.1: Fourier spectrum of ϕ_{1+} when n is even

$j \backslash l$	0	1	2
0	0		
1	α		
2	2α		
$\vdots$	$\vdots$		
$\frac{n-1}{2}$	$\frac{n-1}{2}\,\alpha$	$\frac{n-1}{2}\,\alpha - \beta$	
$\frac{n+1}{2}$	$\frac{n+1}{2}\,\alpha$	$\frac{n+1}{2}\,\alpha - \beta$	
$\vdots$		$\vdots$	
$n-1$		$(n-1)\alpha - \beta$	$(n-1)\alpha - 2\beta$
n		$n\alpha - \beta$	

Table 4.2: Fourier spectrum of ϕ_{1+} when n is odd

The Fourier coefficients of ϕ_{1+} can be obtained easily from these tables (similarly to [13] and [15]). We have the following solution to the Riemann-

Hilbert problem (4.2):

$$\phi_{1+} = \sum_{j=0}^{\left[\frac{n+1}{2}\right]} \Big(-\frac{a}{b}\Big)^j e_{j\alpha} + \sum_{j=\left[\frac{n+1}{2}\right]}^{n-1} p\Big(-\frac{a}{b}\Big)^j \Big(-\frac{c}{b}\Big) e_{j\alpha-\beta} + p\left(-\frac{a}{b}\right)^n \frac{c}{b}\, e_\lambda$$
$$+ p\left(-\frac{a}{b}\right)^{n-1}\left(\frac{c}{b}\right)^2 e_{\lambda-\alpha-\beta} + (p-1)\left(-\frac{a}{b}\right)^{\frac{n-1}{2}}\left(-\frac{c}{b}\right) e_{\frac{\lambda-\alpha-\beta}{2}}\,, \qquad (4.48)$$

$$\phi_{1-} = e_{-\lambda}\phi_{1+} \qquad (4.49)$$

$$\phi_{2+} = -2p\left(-\frac{a}{b}\right)^n c - \left(-\frac{a}{b}\right)^{\left[\frac{n+1}{2}\right]} a e_{\left(\left[\frac{n+1}{2}\right]+1\right)\alpha-\lambda} + p\left(-\frac{a}{b}\right)^{n+1} c\, e_\alpha\,, \qquad (4.50)$$

$$\phi_{2-} = b + \sum_{j=0}^{\left[\frac{n}{2}-1\right]} \Big(-\frac{a}{b}\Big)^j c\, e_{j\alpha-\beta} - \sum_{j=\left[\frac{n+1}{2}\right]}^{n-2} p\left(-\frac{a}{b}\right)^j \frac{c^2}{b} e_{j\alpha-2\beta}$$
$$-(p-1)\left(-\frac{a}{b}\right)^{\frac{n-1}{2}} \frac{c^2}{b} e_{\frac{n-1}{2}\alpha-2\beta} - p\left(-\frac{a}{b}\right)^n \frac{c^3}{ab}\, e_{(n-1)\alpha-3\beta}\,, \qquad (4.51)$$

where $p = \begin{cases} 1\,, & \text{if } n \text{ is even,} \\ 2\,, & \text{if } n \text{ is odd.} \end{cases}$

Having obtained a first solution to the Riemann-Hilbert problem, we will now use the results of Chapter 3 to establish necessary and sufficient conditions for existence of a Wiener-Hopf factorization for the matrix G.

By (4.50) and (4.51), we can see that, for sufficiently big $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$ we have

$$\inf_{\mathbb{C}^+ + i\varepsilon_1} (\,|\phi_{1+}| + |\phi_{2+}|\,) \;>\; 0\,, \qquad (4.52)$$
$$\inf_{\mathbb{C}^- - i\varepsilon_2} (\,|\phi_{1-}| + |\phi_{2-}|\,) \;>\; 0. \qquad (4.53)$$

So, from Theorem 3.11, to prove that $\phi_\pm$ are corona pairs, it is enough to show that one of the conditions (i)-(iv) in Theorem 3.9 holds in a strip $S = S_{-\varepsilon_2,\varepsilon_1}$ for $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$ such that (4.52) and (4.53) hold. Therefore, we just have to show that

$$\inf_S \left(|\phi_{2+}| + \left| \frac{\phi_{2-} - e_\lambda \phi_{2+}}{g} \right| \right) > 0. \qquad (4.54)$$

For a strip S of the above-mentioned type, (4.54) is equivalent to

$$\inf_S \left(|e_\lambda \phi_{2+}| + \left| \frac{\phi_{2-} - e_\lambda \phi_{2+}}{g} \right| \right) > 0.$$

With the notation

$$f = e_\lambda \phi_{2+}\,, \qquad\qquad h = \frac{\phi_{2-} - e_\lambda \phi_{2+}}{g}$$

we have

$$f = F \circ \varphi \qquad \text{and} \qquad h = H \circ \varphi$$

with $\varphi : S \longrightarrow \mathbb{C}^2$ defined by (3.38) and

$$\begin{aligned} F(z,w) &= -2p\left(-\frac{a}{b}\right)^n c z^n w - \left(-\frac{a}{b}\right)^{\left[\frac{n+1}{2}\right]} a z^{\left[\frac{n+1}{2}\right]+1} + p\left(-\frac{a}{b}\right)^{n+1} c\, z^{n+1} w\,, \\ H(z,w) &= \frac{N(z,w)}{D(z,w)}\,, \end{aligned}$$

with

$$\begin{aligned} N(z,w) = b &+ \sum_{j=0}^{\left[\frac{n}{2}-1\right]} \left(-\frac{a}{b}\right)^j c\, z^j w - \sum_{j=\left[\frac{n+1}{2}\right]}^{n-2} p\left(-\frac{a}{b}\right)^j \frac{c^2}{b} z^j w^2 \\ &- (p-1)\left(-\frac{a}{b}\right)^{\frac{n-1}{2}} \frac{c^2}{b} z^{\frac{n-1}{2}} w^2 - p\left(-\frac{a}{b}\right)^n \frac{c^3}{ab}\, z^{n-1} w^3 - F(z,w) \end{aligned}$$

and $D(z,w) = az + b + cw$.

Notice that H is a polynomial given by

$$\begin{aligned} H(z,w) = &\sum_{j=0}^{\left[\frac{n+1}{2}\right]} \left(-\frac{a}{b}\right)^j z^j + \sum_{j=\left[\frac{n+1}{2}\right]}^{n-1} p\left(-\frac{a}{b}\right)^j \left(-\frac{c}{b}\right) z^j w + p\left(-\frac{a}{b}\right)^n \frac{c}{b}\, z^n w \\ &+ p\left(-\frac{a}{b}\right)^{n-1} \left(\frac{c}{b}\right)^2 z^{(n-1)} w^2 + (p-1)\left(-\frac{a}{b}\right)^{\frac{n-1}{2}} \left(-\frac{c}{b}\right) z^{\frac{n-1}{2}} w \end{aligned}$$

as well as F, so that F and H are continuous in $\mathbb{C}^2$.

Following Theorem 3.17, we now study the solutions of

$$\begin{cases} F(z,w) = 0 \\ H(z,w) = 0. \end{cases} \tag{4.55}$$

For any solution of (4.55), we have $w \neq 0$ and $z \neq 0$ and the first equation of (4.55) is equivalent to

$$w^{-1} z^{-\left[\frac{n}{2}-1\right]} = -2p\,\frac{c}{a}\left(-\frac{a}{b}\right)^{\left[\frac{n}{2}\right]} - p\left(\frac{c}{b}\right)\left(-\frac{a}{b}\right)^{\left[\frac{n}{2}\right]} z. \tag{4.56}$$

On the other hand, it is clear that all the solutions to (4.55) must also satisfy

$$\begin{cases} F(z,w) = 0 \\ \\ N(z,w) = 0 \end{cases} \quad \text{with} \quad z \neq 0, \quad w \neq 0. \tag{4.57}$$

Therefore, we start by determining all the solutions to (4.57), which, taking (4.56) into account and using the change of variable $x = \frac{a}{b}\,z$, is equivalent to

$$\begin{cases} w^{-1} x^{-\left[\frac{n}{2}-1\right]} = -p\,\frac{c}{b}\,(-1)^{\left[\frac{n}{2}\right]}(x+2) \\ \\ -p\,(-x)^{\left[\frac{n+4}{2}\right]} + 7p\,(-x)^{\left[\frac{n+2}{2}\right]} - 14p\,(-x)^{\left[\frac{n}{2}\right]} + 8p\,(-x)^{\left[\frac{n}{2}-1\right]} + x + 4 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} w^{-1} x^{-\left[\frac{n}{2}-1\right]} = -p\,\frac{c}{b}\,(-1)^{\left[\frac{n}{2}\right]}(x+2) \\ \\ (x+4)\left(p\,(-x)^{\left[\frac{n+2}{2}\right]} - 3p\,(-x)^{\left[\frac{n}{2}\right]} + 2p\,(-x)^{\left[\frac{n-2}{2}\right]} + 1\right) = 0 \end{cases}$$

(with $x \neq 0,\ w \neq 0$).

For $x = -4$, this system admits a solution given by

$$(z_0, w_0) = \left(-4\,\frac{b}{a}\,, -2^{1-n}\,\frac{b}{c}\right). \tag{4.58}$$

It is clear that $H(z_0, w_0) = 0$ (since $N(z_0, w_0) = 0$, $D(z_0, w_0) \neq 0$).

As to the points (z_j, w_j), $j = 1, \ldots, \left[\frac{n+2}{2}\right]$, which satisfy

$$p\,(-x)^{\left[\frac{n+2}{2}\right]} - 3p\,(-x)^{\left[\frac{n}{2}\right]} + 2p\,(-x)^{\left[\frac{n-2}{2}\right]} + 1 = 0,$$

it is easy to see that $H(z_j, w_j) \neq 0$.

So we conclude that the unique solution to the system (4.55) is (z_0, w_0) given by (4.58).

Let $S = S_{-\varepsilon_2,\varepsilon_1}$ be such that (3.39) is satisfied, as well as (4.52) and (4.53). From Theorem 3.17 we conclude that

$$\inf_S \left(|\phi_{2+}| + \left| \frac{\phi_{2-} - e_\lambda \phi_{2+}}{g} \right| \right) > 0$$

iff

$$\left| 4\,\frac{b}{a} \right|^{\frac{\beta}{\alpha}} \neq \left| 2^{n-1}\,\frac{c}{b} \right|.$$

Taking Theorems 3.2 and 3.4 into account, we have proved the following theorem.

Theorem 4.3. *([9]) Let G be given by (4.1) where g is of the form (4.44), with α, β satisfying (4.46) and (4.47) for $n \geqslant 2$. Then T_G is invertible iff*

$$\left| 4\,\frac{b}{a} \right|^{\frac{\beta}{\alpha}} \neq \left| 2^{n-1}\,\frac{c}{b} \right|. \tag{4.59}$$

Analogously to what was done in the previous example, this case can also be studied throughout the Portuguese Transformation, but this time one application of this method is enough to transform our example into a studied one.

Once again, instead of studying the matrix G in (4.1), with g given by (4.44), we study

$$JG^*J = \begin{bmatrix} e_{-\lambda} & 0 \\ \overline{g} & e_\lambda \end{bmatrix} \tag{4.60}$$

where $\overline{g} = \overline{a}e_{-\alpha} + \overline{b} + \overline{c}e_\beta$.

Since $-\alpha \in \mathrm{sp}(\overline{g})$ we can write

$$e_\alpha \overline{g} = \overline{a} + \overline{b}e_\alpha + \overline{c}e_{\alpha+\beta} \tag{4.61}$$

where

$$q_0 = \overline{a}\,, \quad q_1 = \overline{b}\,, \quad q_2 = \overline{c} \tag{4.62}$$

and

$$\gamma_1 = \alpha\,, \quad \gamma_2 = \alpha + \beta. \tag{4.63}$$

Applying the Portuguese Transformation to the matrix (4.60), we have that this matrix is factorable only simultaneously with

$$\begin{bmatrix} e_{-\alpha} & 0 \\ g_1 & e_{\alpha} \end{bmatrix}. \tag{4.64}$$

where

$$g_1 = \sum_{(n,\gamma)-\lambda<\alpha} q_0^{-1} d_n e_{(n,\gamma)-\lambda}$$

with $(n,\gamma) := n_1\gamma_1 + n_2\gamma_2$ and γ_1, γ_2 are given by (4.63).

Clearly, we have

$$\mathrm{sp}(g_1) \subset \{(n_1+n_2)\alpha + n_2\beta - \lambda \ : \ n_1, n_2 \in \mathbb{Z}^+\} \cap (-\alpha, \alpha). \tag{4.65}$$

It is not difficult to see that all the terms of $\mathrm{sp}(g_1)$ in the interval $(-\alpha,\alpha)$ are of the form (4.65) with: $n_1 = 0\,, n_2 = 1$ or $n_1 = n_2 = 1\,,$ or $n_1 = \left[\frac{n+1}{2}\right]\,, n_2 = 0$ or $n_1 = \left[\frac{n+3}{2}\right]\,, n_2 = 0$.

So it follows that

$$\mathrm{sp}(g_1) \subset \{\, \beta - \left[\frac{n}{2}\right]\alpha\,,\ 2\beta - (n-1)\alpha\,,\ \beta - \left[\frac{n-2}{2}\right]\alpha\,,\ 2\beta - (n-2)\alpha\,\}.$$

Taking into account (4.61)-(4.63), the particular values of n_1, n_2 and (5.16), we can determine the Fourier coefficients of g_1.

Therefore we have

$$g_1 = \frac{1}{\overline{a}}\left(-\frac{\overline{b}}{\overline{a}}\right)^{\left[\frac{n+1}{2}\right]} e_{\beta-\left[\frac{n}{2}\right]\alpha} - \frac{\overline{c}}{\overline{a}^2} e_{2\beta-(n-1)\alpha} + \frac{1}{\overline{a}}\left(-\frac{\overline{b}}{\overline{a}}\right)^{\left[\frac{n+3}{2}\right]} e_{\beta-\left[\frac{n-2}{2}\right]\alpha} + \frac{2\overline{b}\overline{c}}{\overline{a}^3} e_{2\beta-(n-2)\alpha}. \tag{4.66}$$

Since all the coefficients are different from zero, the matrix (4.64) with g_1 given by (4.66) satisfies the conditions of Theorem 3.1 in [26], where a necessary and sufficient condition for the existence of a canonical factorization was established. So the matrix (4.64) (and therefore (4.60)) admits a canonical factorization iff

$$\left|\frac{2\bar{b}\bar{c}}{\bar{a}^3}\right|^{(n-1)\alpha-2\beta}\left|\frac{\bar{c}}{\bar{a}^2}\right|^{2\beta-(n-2)\alpha}\neq\left|\frac{1}{\bar{a}}\left(\frac{\bar{b}}{\bar{a}}\right)^{\left[\frac{n+3}{2}\right]}\right|^{-\beta+\left[\frac{n}{2}\right]\alpha}\left|\frac{1}{\bar{a}}\left(\frac{\bar{b}}{\bar{a}}\right)^{\left[\frac{n+1}{2}\right]}\right|^{\beta-\left[\frac{n-2}{2}\right]\alpha},$$

which, after some computation, is equivalent to condition (4.59).

It should be noticed that the study of the invertibility of Toeplitz operators with symbol G given by (4.1), where g satisfies (4.44)–(4.47), was carried out for the first time in [9]. Afterwards, taking into account the recent paper [26], where a more general case is studied, we see that after one application of the Portuguese Transformation our results concerning factorization can be obtained from this paper.

4.1.4 Some critical remarks

In all these examples we applied two different methods, one of them explained in Chapter 3, the other one being the Portuguese Transformation.

Both methods present advantages and disadvantages, as it is natural to expect, depending on the type of symbols and the goals to be attained (existence criteria or constructive methods).

If, by applying the Portuguese Transformation, we reduce the matrix G that we want to study to another one for which factorability criteria are known, then we can also establish conditions for factorability of G (and the factors for the latter can in principle be determined from those for the former, if these are known). However, this may involve a great amount of computations, even when one is familiar with manipulating the Portuguese Transformation. But in some cases, as Example 4.1.3, it is easier to apply this procedure than the other one.

The preliminary determination of a solution to a Riemann-Hilbert problem, when using the approach of Chapter 3, has the advantage of still giving some information on the solutions of the Riemann-Hilbert problem, even when the matrix G is not factorable, which does not happen with the Portuguese Transformation. Moreover, in case of existence of a factorization, it also provides partial formulas for the factors.

One other advantage of taking the Riemann-Hilbert approach proposed in this thesis is the fact that we can apply it for non-AP symbols and with an infinite number of points in its spectrum (as we will see in the next sections) which does not seem possible with the Portuguese Transformation in its current form.

On the other hand, it should be remarked that the approach adopted in this thesis cannot always be applied. In particular it depends on the possibility of determining an appropriate solution to the Riemann-Hilbert problem (4.2) and verifying if some corona type conditions are satisfied. The same can be said about the Portuguese Transformation, since its usefulness also depends on the possibility of reducing the study of some symbols to previously studied ones.

4.2 An example of invertibility by reduction

In this section we apply the results of Section 3.2 in a different way. We study the invertibility for a class of Toeplitz operators by reducing it to the study of other classes with simpler symbols.

In this example we use the results of [13] as a starting point of our study and we assume thus $\lambda = 1$ in (4.1). In fact we can reduce the study of the invertibility of T_G, with G of the form (4.1) with $\lambda > 0$, to the case where $\lambda = 1$ since $G(\xi)$ admits a canonical factorization only simultaneously with

$$\tilde{G}(\xi') = G\left(\frac{\xi'}{\lambda}\right) = \begin{bmatrix} e^{-i\xi'} & 0 \\ g\left(\frac{\xi'}{\lambda}\right) & e^{i\xi'} \end{bmatrix}$$

(with $\tilde{G}_\pm(\xi') = G_\pm\left(\frac{\xi'}{\lambda}\right)$).

Let $\alpha, \beta \in]0,1[, \frac{\alpha}{\beta} \notin \mathbb{Q}, \alpha + \beta > 1$. We assume that N, which will be used below, is a non-negative integer depending on the values of α and β, defined as follows:

$$N = \min J$$

with

$$J = \left\{ j \in \mathbb{N} \cup \{0\} : \left[\frac{1 + j\beta}{\alpha}\right] = \left[\frac{(j+1)\beta}{\alpha}\right] \text{ or } n\alpha - j\beta = 1 \text{ for some } n \in \mathbb{N} \right\}$$

(where $[X]$ denotes the integer part of the real number X).

Now let g in (4.1) take the form

$$g = ce_{-\beta} + b + \sum_{j=1}^{n} a_j e_{j\alpha} + d_- e_{-\mu} \tag{4.67}$$

where $n = \left[\frac{1}{\alpha}\right]$, if $\frac{1}{\alpha} \notin \mathbb{N}$ and $n = \left[\frac{1}{\alpha}\right] - 1$, if $\frac{1}{\alpha} \in \mathbb{N}$; $d_- \in H_\infty^-$; c, b, a_j ($j = 1, 2, \ldots, n$) are complex numbers with $b \neq 0$, $\sum_{j=1}^{k} |a_j| \neq 0$; $\mu \in]0,1[$ is such that

$$\mu > \max\{S_{l+1}\,\alpha - l\beta : l = 0, 1, \ldots, N\} = M \tag{4.68}$$

with

$$\begin{aligned} S_l &= \left[\frac{1 + (l-1)\beta}{\alpha}\right] \quad \text{for} \quad l = 0, 1, \ldots, N \\ S_{N+1} &= \left[\frac{(N+1)\beta}{\alpha}\right] \end{aligned}$$

(cf. Definition 4.8 in [13]).

So in this case g can be written as

$$g = p + d_- e_{-\mu} \tag{4.69}$$

for

$$p = ce_{-\beta} + b + \sum_{j=1}^{n} a_j e_{j\alpha} \,.$$

The subclass of matrix functions $\tilde{G}$ for which $d_- = 0$ in (4.67) (i.e., $g = p$) was completely studied in [13], where it was shown that $\tilde{G}$ admits a canonical factorization, which was explicitly obtained.

The Riemann-Hilbert problem $\tilde{G}\phi_+ = \tilde{\phi}_-$, which is equivalent to

$$\begin{cases} e_{-1} q_+ = q_- \\ p\, q_+ = e_1 u_+ + u_- \end{cases}, \tag{4.70}$$

admits a non-zero solution of the form $\phi_+ = (q_+, -u_+)$, $\tilde{\phi}_- = (e_{-1}q_+, u_-)$, with $q_+, u_+ \in H_\infty^+$, $u_- \in H_\infty^-$, $\operatorname{sp} q_+ \subset [0,1]$. This solution is known from [13] and, in particular, we have

$$q_+ = \sum_{l=0}^{N} \sum_{j=S_l}^{S_{l+1}} \tilde{A}_{j,l}\; e_{j\alpha - l\beta}, \tag{4.71}$$

(where the Fourier coefficients $\tilde{A}_{j,l}$ are defined in Theorem 4.15 of [13]) and u_+ and u_- are uniquely defined by p and q_+, according to the second equation of (4.70). We remark here that $0 \in \operatorname{sp} u_-$ (cf. [13]).

Taking (4.69) into account, it is clear that

$$G\phi_+ = \tilde{\phi}_- + \begin{bmatrix} 0 \\ d_- e_{-\mu} q_+ \end{bmatrix}. \tag{4.72}$$

Denoting the right-hand side of (4.72) by ϕ_- we have $\phi_- \in (H_\infty^-)^2$, since condition (4.68) implies that μ is greater than $M = \max \operatorname{sp}(q_+)$, as we can see from (4.71).

Since $G, \tilde{G} \in (H_\infty(S_{-\varepsilon_2,0}))^{2\times 2}$ (for any $\varepsilon_2 > 0$), $\gamma = \tilde{\gamma} = 1$, (3.33) is satisfied and $\tilde{G}$ admits a bounded canonical Wiener-Hopf factorization, then it follows from Theorem 3.13 that G admits a canonical Wiener-Hopf factorization if (3.28) holds.

Since, in this case,

$$\phi_{2-} = u_- + d_- e_{-\mu} q_+ = u_- + (d_- e_{-M} q_+) e_{-(\mu - M)}$$

and $d_- e_{-M} q_+ \in H_\infty^-$, $\mu - M > 0$ and $0 \in \mathrm{sp}(u_-)$, we see that, in fact, (3.28) holds (for sufficiently big ε_2) and thus we get the following.

Theorem 4.4. *([9]) Let G take the form (4.1) where g is given by (4.67) and μ satisfies (4.68). Then G admits a canonical Wiener-Hopf factorization.*

Analogously, if we define $\tilde{N}$ and $\tilde{S}_l$ ($l = 0, 1, \ldots, \tilde{N}+1$) in the same way as N and S_l, respectively, with α replaced by β and vice-versa, we have the following result.

Theorem 4.5. *([9]) Let G take the form (4.1) where*

$$g = \sum_{j=1}^{n} c_j e_{-j\beta} + b + a e_\alpha + d_+ e_\mu$$

where $n = \left[\frac{1}{\beta}\right]$, if $\frac{1}{\beta} \notin \mathbb{N}$ and $n = \left[\frac{1}{\beta}\right] - 1$, if $\frac{1}{\beta} \in \mathbb{N}$; $d_+ \in H_\infty^+$; a, b, c_j ($j = 1, 2, \ldots, n$) are complex numbers with $b \neq 0$, $\sum_{j=1}^{k} |c_j| \neq 0$, $\alpha, \beta, \mu \in]0,1[$ with $\frac{\alpha}{\beta} \notin \mathbb{Q}$, $\alpha + \beta > 1$ and

$$\mu > \max\{\tilde{S}_{l+1}\beta - l\alpha : l = 0, 1, \ldots, \tilde{N}\} = \tilde{M}.$$

Then G admits a canonical Wiener-Hopf factorization.

It should be noticed that in this example we are no longer restricted to almost periodic symbols (contrary to what happened in the previous examples of this chapter).

Remark 4.6. *A particular case of matrix functions G satisfying the conditions of the last theorem, with $g = ce_{-\beta}+b+ae_{\alpha}+de_{2\alpha-\beta}$, where $a, b, c, d \in \mathbb{C}$, $b \neq 0$, $\frac{1}{2} \leq \beta < \alpha < 2\alpha - \beta < 1$ and $3\alpha - 2\beta > 1$, was considered in [13], where an explicit (almost-periodic polynomial) canonical Wiener-Hopf factorization was obtained. Indeed we have, in this case, $\tilde{N} = 0$, $\tilde{M} = \beta$ and, taking $\mu = 2\alpha - \beta$, we see that the conditions of Theorem 4.5 are satisfied. As regards the mere question of existence of a canonical Wiener-Hopf factorization, Theorems 4.4 and 4.5 can be seen as a generalization of those obtained in [13]. In fact, we see that as $\tilde{M} = \beta$, conditions of Theorem 4.5 are satisfied for $\mu > \beta$ so we conclude moreover that G admits a canonical Wiener-Hopf factorization for g given by*

$$g = ce_{-\beta} + b + d_+ e_\mu\,, \quad \mu > \beta\,, \quad d_+ \in H^+_\infty\,.$$

It should be remarked that a direct verification of the corona conditions $\phi_\pm \in CP^\pm$ is in general quite difficult, even if $d_\pm$ is an almost-periodic polynomial. For instance, taking

$$g = cEe_{-\beta} + b + \sum_{j=1}^{n} a_j e_{j\alpha} + de_{n\alpha-2\beta}$$

with $d \in \mathbb{C}$, $2\beta - n\alpha < 1$ and $\beta > n\alpha$, it can be easily verified that the conditions of Theorem 4.4 are satisfied (and therefore T_G is invertible), while a direct study of the solutions $\phi_\pm$ of (4.2) would be much more difficult. For instance, for $n = 4$, a solution to (4.2) is given by

$$\phi_{1+} = q_+ = 1 + A_1 e_\alpha + A_2 e_{2\alpha} + A_3 e_{3\alpha} + A_4 e_{4\alpha},$$

$$\begin{aligned}-e_1\phi_{2+} = e_1 u_+ = {}& (a_4A_1+a_3A_2+a_2A_3+a_1A_4)e_{5\alpha}+(a_4A_2+a_3A_3+a_2A_4)e_{6\alpha}+\\ &+(a_4A_3+a_3A_4)e_{7\alpha}+a_4A_4e_{8\alpha},\end{aligned}$$

$$\phi_{1-} = e_{-1}\phi_{1+},$$

$$\phi_{2-} = b + ce_{-\beta} + cA_1 e_{\alpha-\beta} + cA_2 e_{2\alpha-\beta} + cA_3 e_{3\alpha-\beta} + cA_4 e_{4\alpha-\beta} + de_{4\alpha-2\beta} +$$
$$+ dA_1 e_{5\alpha-2\beta} + dA_2 e_{6\alpha-2\beta} + dA_3 e_{7\alpha-2\beta} + dA_4 e_{8\alpha-2\beta}.$$

where

$A_1 = -\frac{a_1}{b} \qquad A_2 = -\frac{a_2}{b} + \frac{a_1^2}{b^2}\,, \qquad A_3 = -\frac{a_3}{b} + 2\frac{a_1 a_2}{b^2} - \frac{a_1^3}{b^3}\,,$

$A_4 = -\frac{a_4}{b} + 2\frac{a_1 a_3}{b^2} + \frac{a_2^2}{b^2} - 3\frac{a_1^2 a_2}{b^3} + \frac{a_1^4}{b^4}.$

4.3 Toeplitz operators with symbols with a gap around zero

In this section, we study the invertibility and Fredholmness of Toeplitz operators T_G where G is a triangular matrix symbol of the form (4.1) with

$$g = a_- e_{-\beta} + a_+ e_\nu\,, \tag{4.73}$$

$a_\pm \in H_\infty^\pm$ and $\nu, \beta \in]0, \lambda[$, using the notation established in (2.1). Let

$$g_- = a_- e_{-\beta} \quad \text{and} \quad g_+ = a_+ e_\nu. \tag{4.74}$$

The results of this section are a development of those in [10].

Once again, we approach this problem by studying a solution to a Riemann-Hilbert problem of the form (4.2). Before proceeding, let us state a result which will be useful in this section. It should be remarked that this result holds for any matrix function G with entries in $L_\infty(\mathbb{R})$ without imposing any condition on the size of the matrix or on its determinant.

Theorem 4.7. *([10]) Let $G \in (L_\infty(\mathbb{R}))^{2\times 2}$ and (ϕ_+, ϕ_-) be a non trivial solution to (4.2) such that one of the following conditions is satisfied for some $\gamma > 0$:*

(i) $\phi_+ = e_\gamma\, \phi_+^*$ *with* $\phi_+^* \in (H_\infty^+)^2$,

(ii) $\phi_- = e_{-\gamma}\,\phi_-^*$ *with* $\phi_-^* \in (H_\infty^-)^2$,

where e_γ *and* $e_{-\gamma}$ *are defined according to (2.1). Then* $\dim\ker T_G = \infty$, T_G *is not Fredholm and* G *does not admit a Wiener-Hopf factorization.*

Proof. Let, for instance, (i) hold. Then for any $\mu \in]0,\gamma]$ we have $\phi_+ = e_\mu(e_{\gamma-\mu}\,\phi_+^*) = e_\mu\,\phi_{0+}$ with $\phi_{0+} \in (H_\infty^+)^2$ and, from (4.2),

$$G(e_\mu \phi_{0+}) = \phi_-\,, \qquad G\,\phi_{0+} = e_{-\mu}\,\phi_-.$$

Thus, defining

$$\psi_+(\xi) = \frac{e_\mu - 1}{\xi}\,\phi_{0+}, \qquad \psi_-(\xi) = \frac{1 - e_{-\mu}}{\xi}\,\phi_-,$$

we have $G\psi_+ = \psi_-$ with $\psi_\pm \in (H_p^\pm)^2$ and we conclude that there are infinitely many linearly independent elements in $\ker T_G$, so T_G is not Fredholm.

The proof is similar if (ii) holds, since it is clear that (ii) can be reduced to (i). ∎

So taking into account this theorem and the results of the previous chapter, we will study, in this case, the Fredholmness and invertibility of T_G and therefore the existence of a Wiener-Hopf factorization of the symbol G.

If, in particular, $a_\pm$ are $APW^\pm$ functions, we establish conditions for the existence of an APW factorization of the symbol G and, if it exists, we also determine its AP partial indices. Finally, we consider particular subclasses of matrix symbols where these conditions for existence of an APW factorization present a higher degree of explicitness.

We remark that even when g is an APW function, since $a_\pm$ can have an infinite number of points in their spectrum, the Portuguese Transformation may be extremely difficult, or even impossible, to apply.

4.3.1 The class $\mathcal{S}_{\frac{\lambda}{N}}$

If g can be written in the form (4.73) for a certain pair of values $(\nu, \beta) \in]0, \lambda[^2$, then it is clear that it can also be written in the form $g = a_-^* e_{-\beta^*} + a_+^* e_{\nu^*}$ with $a_\pm^* \in H_\infty^\pm$ for infinitely many other pairs $(\nu^*, \beta^*) \in]0, \lambda[^2$.

We define

$$L(g) = \{(\nu^*, \beta^*) \in]0, \lambda[^2 \colon g = a_-^* e_{-\beta^*} + a_+^* e_{\nu^*} \quad \text{with} \quad a_\pm^* \in H_\infty^\pm\} \tag{4.75}$$

and

$$N_g = \min_{(\nu^*, \beta^*) \in L(g)} \left\lceil \frac{\lambda}{\nu^* + \beta^*} \right\rceil , \tag{4.76}$$

where $\lceil x \rceil$ denotes the smallest integer which is greater or equal to x.

Since $\nu^* + \beta^* < 2\lambda$, we have $N_g \geq 1$ and there is $(\nu, \beta) \in L(g)$ such that

$$N_g = \left\lceil \frac{\lambda}{\nu + \beta} \right\rceil \tag{4.77}$$

(so that either $\nu + \beta \geq \lambda$ or $\nu + \beta \in \left[\frac{\lambda}{N_g}, \frac{\lambda}{N_g - 1}\right[$) and g can be written as in (4.73).

It is clear that, multiplying a_- and a_+ by appropriate exponentials, we can write g in the form

$$g = \tilde{a}_- e_{-\tilde{\beta}} + \tilde{a}_+ e_{\tilde{\nu}} \qquad \text{with} \qquad \tilde{\nu} + \tilde{\beta} = \tfrac{\lambda}{N_g}$$

(taking, for instance, $\tilde{\nu} = \nu,\ \tilde{a}_+ = a_+,\ \tilde{\beta} = \frac{\lambda}{N_g} - \nu,\ \tilde{a}_- = e_{-(\beta - \tilde{\beta})} a_-$). Thus we will assume in what follows that g takes the form (4.73) with

$$\nu + \beta = \frac{\lambda}{N_g}. \tag{4.78}$$

Let us assume moreover the following notation.

Definition 4.8. *Let $x, y \in \mathbb{R}$ and $f \in L_\infty(\mathbb{R})$. We say that*

(i) $\operatorname{sp}(f) \subset [x, +\infty[\,$, *(ii)* $\operatorname{sp}(f) \subset]-\infty, x]\,$, *(iii)* $\operatorname{sp}(f) \subset [x, y]$

iff

(i') $e_{-x}f \in H_\infty^+$, (ii') $e_{-x}f \in H_\infty^-$, (iii') $e_{-y}f \in H_\infty^-$ and $e_{-x}f \in H_\infty^+$, *respectively.*

Given $N \geq 1$, we denote by $S_{\frac{\lambda}{N}}$ the class of functions g of the form (4.73), such that $N_g = N$ and

$$\operatorname{sp}(a_-) \subset \left[-\frac{\beta}{N-1}, 0\right], \quad \operatorname{sp}(a_+) \subset \left[0, \frac{\nu}{N-1}\right], \tag{4.79}$$

where (ν, β) are such that (4.73) and (4.78) hold. By $\mathcal{S}_{\frac{\lambda}{N}}$ we denote the class of 2×2 matrix functions G of the form (4.1) with $g \in S_{\frac{\lambda}{N}}$.

Since g is the sum of two functions, $g = g_- + g_+$, where $g_\pm$ are given by (4.74), we have

$$\operatorname{sp}(g_-) \subset \left[-\frac{N\beta}{N-1}, -\beta\right], \qquad \operatorname{sp}(g_+) \subset \left[\nu, \frac{N\nu}{N-1}\right], \tag{4.80}$$

which accounts for denoting symbols $G \in \mathcal{S}_{\frac{\lambda}{N}}$ by "symbols with a gap around zero".

In order to apply the previous results to Toeplitz operators with symbols in this class, we start by determining a solution to (4.2).

For $N = 1$ such a solution is given by

$$\phi_+ = (e_\beta, -a_+), \qquad \phi_- = (e_{-\nu}, a_-) \tag{4.81}$$

and can be checked directly.

Thus taking this solution to the Riemann-Hilbert problem and applying Theorems 3.2 and 4.7 we get the following result which establishes conditions for Fredholmness and invertibility of T_G.

Theorem 4.9. *([10]) Let $G \in \mathcal{S}_\lambda$.*

(i) If there is $\alpha_1 \in \mathbb{R}^+$ such that

$$e_{-\alpha_1}a_+ \in H_\infty^+ \quad \text{or} \quad e_{\alpha_1}a_- \in H_\infty^-,$$

then T_G is not Fredholm, since $\dim\ker T_G = \infty$.

(ii) T_G is invertible if there exist $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$ such that

$$\inf_{\xi \in \mathbb{C}^+ + i\varepsilon_1} |a_+(\xi)| > 0\,, \qquad \inf_{\xi \in \mathbb{C}^- - i\varepsilon_2} |a_-(\xi)| > 0\,. \tag{4.82}$$

Proof. (i) is a consequence of Theorem 4.7.

(ii) It is enough to see that (4.82) is equivalent to

$$\inf_{\mathbb{C}^+}(\,|e_{\lambda-\nu}| + |a_+|\,) > 0\,, \qquad \inf_{\mathbb{C}^-}(\,|e_{-\nu}| + |a_-|\,) > 0\,,$$

which means that $\phi_\pm \in CP^\pm$, for $\phi_\pm$ given by (4.81). Therefore Theorem 3.2 implies that T_G is invertible. ■

Remark 4.10. *We can interpret the first part of this result as saying that, for T_G (with $G \in (APW)^{2\times 2}$ satisfying the assumptions of Theorem 4.9) to be Fredholm, it is necessary that the "length of the gap" in* $\operatorname{sp} g$ *is not greater than λ. This generalizes a similar result regarding invertibility of T_G in [12] (cf. Theorem 4.6).*

In the case where $N > 1$ in (4.78), a solution to (4.2) can be determined from (4.81) taking into account that

$$(a_+ e_\nu)^N + (a_- e_{-\beta})^N = a_+^N e_{N\nu} + a_-^N e_{-N\beta}$$

where, if (4.78) is satisfied,

$$N\nu + N\beta = \lambda$$

and, on the other hand,

$$(x^N + (-1)^{N-1} y^N) = (x+y) \sum_{j=0}^{N-1} \left((-1)^j x^{N-1-j}\, y^j\right). \tag{4.83}$$

Taking

$$x = a_+ e_\nu \,, \qquad y = a_- e_{-\beta} \,,$$

we have

$$\begin{bmatrix} e_{-\lambda} & 0 \\ x^N + (-1)^{N-1} y^N & e_\lambda \end{bmatrix} \begin{bmatrix} e_{N\beta} \\ -a_+^N \end{bmatrix} = \begin{bmatrix} e_{-N\nu} \\ (-1)^{N-1} a_-^N \end{bmatrix}$$

and thus, from (4.83),

$$\begin{bmatrix} e_{-\lambda} & 0 \\ x+y & e_\lambda \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_{N\beta} \\ -a_+^N \end{bmatrix} = \begin{bmatrix} \Sigma & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_{-N\nu} \\ (-1)^{N-1} a_-^N \end{bmatrix}$$

where

$$\Sigma = \sum_{j=0}^{N-1} \left((-1)^j x^{N-1-j} \, y^j \right) \,.$$

Thus we have the following.

Theorem 4.11. *([10]) Let $G \in \mathcal{S}_{\frac{\lambda}{N}}, N > 1$. Then a solution to the Riemann-Hilbert problem (4.2) is given by $\phi_\pm = (\phi_{1\pm}, \phi_{2\pm})$, where*

$$\phi_{1+} = e_{\lambda-\nu} \sum_{j=0}^{N-1} \left((-1)^j a_+^{N-1-j} a_-^j \, e_{-j\frac{\lambda}{N}} \right) \quad , \quad \phi_{2+} = -a_+^N \,, \tag{4.84}$$

$$\phi_{1-} = e_{-\lambda} \phi_{1+} \quad , \quad \phi_{2-} = (-1)^{N-1} a_-^N . \tag{4.85}$$

Proof. Since we have $G\phi_+ = \phi_-$ and, obviously, $\phi_{2\pm} \in H_\infty^\pm$, it is left to prove that $\phi_{1\pm} \in H_\infty^\pm$. From (4.79) it follows that

$$e_{\frac{\beta}{N-1}} a_- = \tilde{a}_+ \in H_\infty^+$$

so that

$$\phi_{1+} = \sum_{j=0}^{N-1} \left((-1)^j a_+^{N-1-j} \, \tilde{a}_+^j \, e_{\beta - j\frac{\beta}{N-1}} \, e_{\lambda-(j+1)\frac{\lambda}{N}} \right) \in H_\infty^+ \,.$$

Analogously, from

$$e_{-\frac{\nu}{N-1}} a_+ = \tilde{a}_- \in H_\infty^-$$

we have

$$\phi_{1-} = \sum_{j=0}^{N-1} \left((-1)^j \, \tilde{a}_-^{N-1-j} \, a_-^j \, e_{-j(\frac{\nu}{N-1}+\frac{\lambda}{N})} \right) \in H_\infty^- .$$

■

Analogously to what was done when $N = 1$, now we will study this solution to the Riemann-Hilbert problem in order to get some conclusions about the Fredholmness and invertibility of some classes of Toeplitz operators, applying the results of the previous chapters.

As it was mentioned before, to study the invertibility of T_G, we will verify if $\phi_\pm$ are corona pairs in $\mathbb{C}^\pm$, respectively (cf. Theorem 3.2), which in this case seems to be difficult due to their expressions. However, here we will take advantage of the results of Section 3.2 taking into account that all elements in G are entire functions.

For this class of matrix symbol G, the behaviour of $\phi_\pm$ "at infinity", which is translated by (3.27) and (3.28) of Theorem 3.11, is not difficult to study, for big enough $\varepsilon_1, \varepsilon_2 > 0$. Therefore, from this theorem, we are reduced to studying the behaviour of these functions, $\phi_\pm$, in a strip of the complex plane. But, in this case, in a strip this study can be reduced to the study of the behaviour of a_+ and a_-, as it will be shown in the next theorem.

Theorem 4.12. *Let the assumptions of Theorem 4.11 be satisfied and ϕ_{1+}, ϕ_{2+} be given by (4.84). Then for any strip $S = \{\xi \in \mathbb{C} : -\varepsilon_2 < \mathrm{Im}(\xi) < \varepsilon_1\}$, with $\varepsilon_1, \varepsilon_2 \in [\,0, +\infty\,[$ we have*

$$\inf_{\xi \in S}(\,|\phi_{1+}(\xi)| + |\phi_{2+}(\xi)|\,) > 0 \quad \textit{iff} \quad \inf_{\xi \in S}(\,|a_+(\xi)| + |a_-(\xi)|\,) > 0 \,.$$

Proof. We will show that

$$\inf_{\xi \in S}(\,|a_+(\xi)| + |a_-(\xi)|\,) = 0 \quad \text{iff} \quad \inf_{\xi \in S}(\,|\phi_{1+}(\xi)| + |\phi_{2+}(\xi)|\,) = 0 \,.$$

Suppose first that

$$\inf_{\xi\in S}(\,|a_+(\xi)|+|a_-(\xi)|\,)=0.$$

Then there is a sequence $(\xi_n)_{n\in\mathbb{N}}$ with $\xi_n\in S$ such that $a_+(\xi_n)\to 0$ and $a_-(\xi_n)\to 0$. So taking into account the expressions of ϕ_{1+},ϕ_{2+} given by (4.84), we must have $\phi_{1+}(\xi_n)\to 0$ and $\phi_{2+}(\xi_n)\to 0$. Therefore

$$\inf_{\xi\in S}(\,|\phi_{1+}(\xi)|+|\phi_{2+}(\xi)|\,)=0.$$

Conversely, if we have

$$\inf_{\xi\in S}(\,|\phi_{1+}(\xi)|+|\phi_{2+}(\xi)|\,)=0,$$

then for some sequence (ξ_n) with $\xi_n\in S$ for all $n\in\mathbb{N}$, we have $\phi_{1+}(\xi_n)\to 0$ and $\phi_{2+}(\xi_n)\to 0$. Thus, from the expression of ϕ_{2+} given by (4.84), it follows that $a_+(\xi_n)\to 0$. From the expression of ϕ_{1+} in (4.84), we can write

$$a_-^{N-1}=(-1)^{N-1}e_{\nu-\frac{\lambda}{N}}\phi_{1+}+(-1)^N e_{\frac{N-1}{N}\lambda}\sum_{j=0}^{N-2}\left((-1)^j a_+^{N-1-j}a_-^j\,e_{-j\frac{\lambda}{N}}\right).$$

Since $\phi_{1+}(\xi_n)\to 0$ and $a_+(\xi_n)\to 0$ we conclude that $a_-(\xi_n)\to 0$ and therefore we get

$$\inf_{\xi\in S}(\,|a_+(\xi)|+|a_-(\xi)|\,)=0\,.$$

■

Since all elements in G are entire functions and $\det G=1$, from Theorem 3.9 we can get a similar conclusion for ϕ_{1-} and ϕ_{2-}.

Corollary 4.13. *Let the assumptions of Theorem 4.11 be satisfied and ϕ_{1-},ϕ_{2-} be given by (4.85). Then for any strip $S=\{\xi\in\mathbb{C}:-\varepsilon_2<\mathrm{Im}(\xi)<\varepsilon_1\}$, with $\varepsilon_1,\varepsilon_2\in[\,0,+\infty\,[$ we have*

$$\inf_{\xi\in S}(\,|\phi_{1-}(\xi)|+|\phi_{2-}(\xi)|\,)>0\quad \textit{iff}\quad \inf_{\xi\in S}(\,|a_+(\xi)|+|a_-(\xi)|\,)>0\,.$$

It should be noticed that even the behaviour of a_+ and a_- in a strip may be difficult to study, but it is clear from the expressions (4.84) and (4.85) that this study is much simpler than studying the behaviour of ϕ_{1+}, ϕ_{2+} or ϕ_{1-}, ϕ_{2-} in a strip.

Taking this into account, let us now establish conditions for invertibility and Fredholmness of Toeplitz operators.

Theorem 4.14. *Let the assumptions of Theorem 4.11 be satisfied.*

(i) If there is $\alpha_1 \in \mathbb{R}^+$ such that either

$$e_{-\alpha_1} a_+ \in H_\infty^+ \ \text{ and } \ \operatorname{sp} a_- \subset [-\beta_1, 0] \ \text{ with } \ 0 < \beta_1 < \frac{\beta}{N-1}\,, \tag{4.86}$$

or

$$e_{\alpha_1} a_- \in H_\infty^- \ \text{ and } \ \operatorname{sp} a_+ \subset [0, \nu_1] \ \text{ with } \ 0 < \nu_1 < \frac{\nu}{N-1}\,, \tag{4.87}$$

then $\dim \ker T_G = \infty$*, so T_G is not Fredholm.*

(ii) T_G is invertible if

$$\inf_{\xi \in S} (\,|a_+(\xi)| + |a_-(\xi)|\,) > 0 \tag{4.88}$$

for any strip $S = \{\xi \in \mathbb{C} : -\varepsilon_2 < \operatorname{Im}(\xi) < \varepsilon_1\}$, with $\varepsilon_1, \varepsilon_2 \in [\,0, +\infty\,[$, and one of the following conditions holds, for $k_1, k_2 \in \mathbb{C}\backslash\{0\}$:

$$(a) \qquad \inf_{\operatorname{Im}(\xi) > \varepsilon_1} |a_+(\xi)| > 0\,, \qquad \inf_{\operatorname{Im}(\xi) < -\varepsilon_2} |a_-(\xi)| > 0\,; \tag{4.89}$$

$$(b) \qquad a_- = k_1\, e_{-\frac{\beta}{(N-1)}} + \tilde{a}_-\,, \quad \text{where} \quad \operatorname{sp} \tilde{a}_- \subset [-\beta_1, 0] \tag{4.90}$$

with $0 < \beta_1 < \frac{\beta}{(N-1)}$ *and*

$$a_+ = k_2\, e_{\frac{\nu}{(N-1)}} + \tilde{a}_+\,, \quad \text{where} \quad \operatorname{sp} \tilde{a}_+ \subset [0, \nu_1] \tag{4.91}$$

with $0 < \nu_1 < \frac{\nu}{(N-1)}$ *;*

$$(c) \qquad a_- \text{ satisfies (4.90) } \ \text{with} \quad \inf_{\operatorname{Im}(\xi) < -\varepsilon_2} |\tilde{a}_-(\xi)| > 0\,; \tag{4.92}$$

$$(d) \qquad a_+ \textit{ satisfies (4.91) with } \inf_{\mathrm{Im}(\xi)>\varepsilon_1} |\tilde{a}_+(\xi)| > 0. \tag{4.93}$$

Proof. (i) Suppose that there is $\alpha_1 \in \mathbb{R}^+$ such that (4.86) holds. Then $a_- = e_{-\beta_1}\tilde{a}_+$ and $a_+ = e_{\alpha_1}A_+$ with $\tilde{a}_+, A_+ \in H_\infty^+$. From (4.84) it follows that

$$\phi_{1+} = e_\tau \tilde{\phi}_{1+} \tag{4.94}$$

where

$$\begin{aligned} \tau &= \beta - (N-1)\beta_1 > 0 \\ \tilde{\phi}_{1+} &= \sum_{j=0}^{N-1} \left((-1)^j A_+^{N-1-j} \tilde{a}_+^j e_{(N-1-j)(\nu+\beta+\alpha_1+\beta_1)} \right) \in H_\infty^+ . \end{aligned}$$

On the other hand, we also have

$$\phi_{2+} = -e_{N\alpha_1} A_+^N. \tag{4.95}$$

From (4.94), (4.95) and Theorem 4.7 we conclude that T_G is not Fredholm.

An analogous proof can be carried out if (4.87) holds.

(ii) From (4.88) and Theorem 4.12 we have

$$\inf_{\xi \in S} (|\phi_{1+}(\xi)| + |\phi_{2+}(\xi)|) > 0 \,, \tag{4.96}$$

for any strip $S = \{\xi \in \mathbb{C} : -\varepsilon_2 < \mathrm{Im}(\xi) < \varepsilon_1\}$, with $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$.

Taking into account that all elements in G are entire functions, it follows from Theorem 3.11 that to prove that $\phi_\pm$ are corona pairs and therefore T_G is invertible, it is enough to study the behaviour of $\phi_{1\pm}, \phi_{2\pm}$ “at infinity”, i.e.,

$$\inf_{\mathrm{Im}(\xi)>\varepsilon_1} (|\phi_{1+}(\xi)| + |\phi_{2+}(\xi)|) \;>\; 0 \,, \tag{4.97}$$

$$\inf_{\mathrm{Im}(\xi)<-\varepsilon_2} (|\phi_{1-}(\xi)| + |\phi_{2-}(\xi)|) \;>\; 0, \tag{4.98}$$

for sufficiently big $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$ such that (4.96) hold. Let us show that these conditions hold in each one of the four cases (a)–(d).

(a) If a_+ and a_- satisfy (4.89), taking into account the expressions of ϕ_{2+} and ϕ_{2-} given by (4.84) and (4.85), respectively, we see that, for sufficiently big $\varepsilon_1, \varepsilon_2 \in [0, +\infty[$, we have

$$\inf_{\mathrm{Im}(\xi) > \varepsilon_1} |\phi_{2+}(\xi)| > 0\,, \qquad \inf_{\mathrm{Im}(\xi) < -\varepsilon_2} |\phi_{2-}(\xi)| > 0\,,$$

and therefore (4.97) and (4.98) hold.

(b) If a_- can be written in the form (4.90) then, from (4.84), we can write ϕ_{1+} in the form

$$\phi_{1+} = C_1 + e_{\gamma_1}\tilde{\phi}_{1+} \quad \text{with} \;\; C_1 = k_1^{N-1}(-1)^{N-1} \neq 0\,,\;\; \gamma_1 > 0\,,\;\; \tilde{\phi}_{1+} \in H_\infty^+ \tag{4.99}$$

and therefore, for sufficiently big $\varepsilon_1 \in [0, +\infty[$, it follows

$$\inf_{\mathrm{Im}(\xi) > \varepsilon_1} |\phi_{1+}(\xi)| > 0. \tag{4.100}$$

Moreover, from the particular form of a_+ given by (4.91) we have $\phi_{1-} = C_2 + e_{-\gamma_2}\tilde{\phi}_{1-}$ with $C_2 = k_2^{N-1} \neq 0$, $\gamma_2 > 0$, $\tilde{\phi}_{1-} \in H_\infty^-$. Therefore

$$\inf_{\mathrm{Im}(\xi) < -\varepsilon_2} |\phi_{1-}(\xi)| > 0,$$

for sufficiently big $\varepsilon_2 \in [0, +\infty[$.

Taking this into account, we conclude that (4.97) and (4.98) hold.

(c) If a_- satisfies (4.92), we can write ϕ_{1+} in the form (4.99) and therefore (4.100) hold. Since

$$\inf_{\mathrm{Im}(\xi) < -\varepsilon_2} |\tilde{a}_-(\xi)| > 0,$$

we have from (4.85) that

$$\inf_{\operatorname{Im}(\xi)<-\varepsilon_2} |\phi_{2-}(\xi)| > 0$$

for sufficiently big $\varepsilon_2 \in [0, +\infty[$ and therefore (4.97) and (4.98) hold.

(d) If a_+ satisfies (4.93), using a similar reasoning we also conclude that (4.97) and (4.98) hold.

■

Remark 4.15. *In this theorem, (i) admits an interpretation similar to that of Remark 4.10, which regarded the case where $\nu + \beta = \lambda$, as long as we can exclude both points $\frac{-N\beta}{(N-1)}$ and $\frac{N\nu}{(N-1)}$ from the spectrum of g (in the sense that* $\operatorname{sp}(g)$ *is contained in a closed interval to which those points do not belong, according to the conditions imposed by (4.86) and (4.87) on* $\operatorname{sp}(a_-)$ *and* $\operatorname{sp}(a_+)$*). In this case, thus, the "length of the gap" in* $\operatorname{sp} g$ *should not be greater than λ/N for T_G to be Fredholm.*

Contrary to what happens in the case $N = 1$, when $N > 1$ it is possible to have invertibility of T_G when the "length of the gap" in $\operatorname{sp} g$ *is greater than λ/N; this happens for instance in (ii)-(b) of the previous theorem.*

Regarding (ii)-(a), it is easy to see that there are $\varepsilon_1, \varepsilon_2 > 0$ such that (4.89) holds when $a_\pm$ take the form $a_+ = k_1 + e_{\gamma_1}\tilde{a}_+, \quad a_- = k_2 + e_{-\gamma_2}\tilde{a}_-$ with $k_1, k_2 \in \mathbb{C}\backslash\{0\}, \quad \gamma_1, \gamma_2 > 0$ and $\tilde{a}_\pm \in H^\pm_\infty$.

Remark 4.16. *If the Toeplitz operator with symbol G is defined in $(H_2^+)^2$, then* $\dim\ker T_G = \dim\operatorname{coker} T_G$ *(cf. [6]) so that, in fact, the results of Theorem 4.9 (i) and 4.14 (i) imply the stronger conclusion that T_G is not semi-Fredholm.*

4.3.2 APW factorization for some particular classes

Let us now consider the case of matrix functions in $\mathcal{S}_{\frac{\lambda}{N}}$, with $N \geq 1$, for which $\mathrm{sp}(g_\pm)$ contain the corresponding endpoints. Using Theorem 3.5, we will show that G in this class always admits an APW factorization and give formulas for its AP partial indices.

Let us consider

$$g = g_- + g_+ \,, \qquad g_\pm \in APW^\pm \tag{4.101}$$

such that

$$\mathrm{sp}(g_-) \subset [-\eta_{2-}, -\eta_{1-}] \quad \text{with} \quad -\eta_{1-}, -\eta_{2-} \in \mathrm{sp}\, g_- \,, \tag{4.102}$$

$$\mathrm{sp}(g_+) \subset [\eta_{1+}, \eta_{2+}] \quad \text{with} \quad \eta_{1+}, \eta_{2+} \in \mathrm{sp}\, g_+ \,, \tag{4.103}$$

with $\eta_{2\pm} \geq \eta_{1\pm} > 0$.

Theorem 4.17. *Let G be a matrix function of the form (4.1), where g is such that (4.101)–(4.103) hold. Then g is in $S_{\frac{\lambda}{N}}$ with*

$$N = \left\lceil \frac{\lambda}{\eta_{1+} + \eta_{1-}} \right\rceil \tag{4.104}$$

if and only if $\eta_{1\pm}$, $\eta_{2\pm}$ satisfy the following conditions:

$$\eta_{1-} \geq \left(\frac{N-1}{N} \right) \eta_{2-} \,, \tag{4.105}$$

$$\eta_{1+} \geq \left(\frac{N-1}{N} \right) \eta_{2+} \,, \tag{4.106}$$

$$\eta_{2+} + \eta_{2-} \leq \frac{\lambda}{N-1} \,. \tag{4.107}$$

Proof. Let us consider that $g \in S_{\frac{\lambda}{N}}$ with $N = \left\lceil \frac{\lambda}{\eta_{1+}+\eta_{1-}} \right\rceil$. In this case, taking into account (4.78) and (4.80), we have

$$\mathrm{sp}(g_-) \subset \left[\frac{-\lambda + N\nu}{N-1} \,, -\frac{\lambda}{N} + \nu \right] , \qquad \mathrm{sp}(g_+) \subset \left[\nu \,, \frac{N\nu}{N-1} \right] .$$

Therefore, from (4.102) and (4.103) we must have

$$\begin{aligned}
-\eta_{2-} &\geq \frac{-\lambda + N\nu}{N-1}\,, \\
-\eta_{1-} &\leq -\frac{\lambda}{N} + \nu\,, \\
\eta_{1+} &\geq \nu\,, \\
\eta_{2+} &\leq \frac{N\nu}{N-1}\,.
\end{aligned}$$

These conditions are equivalent to

$$\nu \leq \frac{\lambda - (N-1)\eta_{2-}}{N}\,, \tag{4.108}$$

$$\nu \geq \frac{\lambda}{N} - \eta_{1-}\,, \tag{4.109}$$

$$\nu \leq \eta_{1+}\,, \tag{4.110}$$

$$\nu \geq \frac{N-1}{N}\eta_{2+}\,. \tag{4.111}$$

From (4.108) and (4.109) we have that (4.105) holds, from (4.110) and (4.111) it follows (4.106) and moreover (4.108) and (4.111) implies (4.107).

Before proving the converse, we remark that (4.108)–(4.111) imply that

$$M_a = \max\left\{\frac{\lambda}{N} - \eta_{1-}, \frac{N-1}{N}\eta_{2+}\right\} \leq \nu \leq \min\left\{\eta_{1+}, \frac{\lambda - (N-1)\eta_{2-}}{N}\right\} = M_i\,. \tag{4.112}$$

Let us now assume that (4.105)–(4.107) are satisfied. Then we have $M_a \leq M_i$. In fact, if $M_a = \frac{\lambda}{N} - \eta_{1-}$, then from (4.104) it follows $\frac{\lambda}{N} - \eta_{1-} \leq \eta_{1+}$ and from (4.105) we have $\frac{\lambda}{N} - \eta_{1-} \leq \frac{\lambda-(N-1)\eta_{2-}}{N}$. If $M_a = \frac{N-1}{N}\eta_{2+}$, from (4.106) we have $\frac{N-1}{N}\eta_{2+} \leq \eta_{1+}$ and we also have $\frac{N-1}{N}\eta_{2+} \leq \frac{\lambda-(N-1)\eta_{2-}}{N}$ from (4.107).

Thus there exists ν such that (4.112) holds. For any ν such that (4.112) is satisfied, we have

$$g = e_{-\frac{\lambda}{N}+\nu}a_- + e_\nu a_+$$

with $a_- = e_{\frac{\lambda}{N}-\nu}g_-$ and $a_+ = e_{-\nu}g_+$ satisfying (4.79). As a matter of fact, from (4.102) and (4.103) we have $\mathrm{sp}(a_-) \subset [\,-\eta_{2-} + \frac{\lambda}{N} - \nu\,, -\eta_{1-} + \frac{\lambda}{N} - \nu\,]$

and $\mathrm{sp}(a_+) \subset [\,\eta_{1+} - \nu\,, \eta_{2+} - \nu\,]$, which satisfy (4.79) if

$$\begin{aligned} -\eta_{2-} + \frac{\lambda}{N} - \nu &\geq -\frac{\beta}{N-1}\,, \\ -\eta_{1-} + \frac{\lambda}{N} - \nu &\leq 0\,, \\ \eta_{1+} - \nu &\geq 0\,, \\ \eta_{2+} - \nu &\leq \frac{\nu}{N-1}\,. \end{aligned}$$

It can be easily seen, taking into account that $\beta = \frac{\lambda}{N} - \nu$, that these inequalities are equivalent to

$$\begin{aligned} \nu &\leq \frac{\lambda - (N-1)\eta_{2-}}{N}\,, \\ \nu &\geq \frac{\lambda}{N} - \eta_{1-}\,, \\ \nu &\leq \eta_{1+}\,, \\ \nu &\geq \frac{N-1}{N}\eta_{2+}\,, \end{aligned}$$

which are verified due to (4.112). ■

Taking into account (4.81) and Theorem 4.11, a solution to the Riemann-Hilbert problem (4.2) can be written in the form

$$\phi_{1+} = e_{\lambda-\nu} \sum_{j=0}^{N-1} \Big((-1)^j (e_{-\nu} g_+)^{N-1-j} (e_{\frac{\lambda}{N}-\nu} g_-)^j \, e_{-j\frac{\lambda}{N}} \Big) \tag{4.113}$$

$$\phi_{2+} = -(e_{-\nu} g_+)^N \tag{4.114}$$

$$\phi_{1-} = e_{-\lambda} \phi_{1+} \tag{4.115}$$

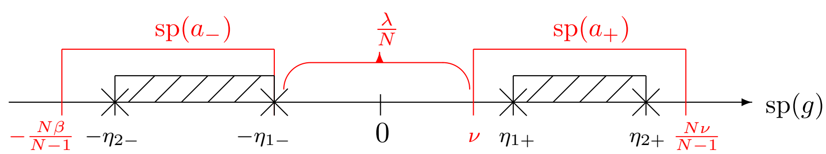
$$\phi_{2-} = (-1)^{N-1} (e_{\frac{\lambda}{N}-\nu} g_-)^N. \tag{4.116}$$

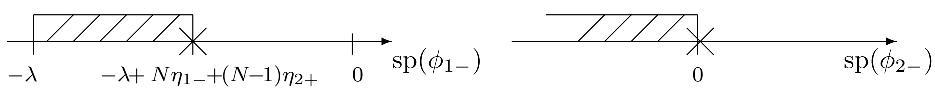
From (4.112) we see that we have to consider two different cases: when $M_a = \frac{\lambda}{N} - \eta_{1-}$, which corresponds to $N\eta_{1-} + (N-1)\eta_{2+} \leq \lambda$, and when $M_a = \frac{N-1}{N}\eta_{2+}$, which means that $N\eta_{1-} + (N-1)\eta_{2+} \geq \lambda$. In each case, there are a lot of possibilities to choose ν in the interval (4.112). Since we will apply Theorem 3.5 to establish conditions for existence of an APW factorization of G, we must have $\phi_- \in CP^-$, so we will choose ν in such a way that $0 \in \mathrm{sp}(\phi_{1-})$ or $0 \in \mathrm{sp}(\phi_{2-})$. In both cases, ϕ_- does not approach zero "at infinity" and therefore to check if $\phi_- \in CP^-$ we just have to study the behaviour of ϕ_- in a strip of the complex plane.

In the first case, $N\eta_{1-} + (N-1)\eta_{2+} \leq \lambda$, we choose $\nu = \frac{\lambda}{N} - \eta_{1-}$. Since $-\eta_{1-} \in \mathrm{sp}(g_-)$ and from (4.116), it is easy to see that

$$\max \mathrm{sp}(\phi_{2-}) = \frac{\lambda}{N} - \nu - \eta_{1-} = 0. \tag{4.117}$$

The next figures illustrate how this choice of ν influences $\mathrm{sp}(\phi_{1-})$ and $\mathrm{sp}(\phi_{2-})$.



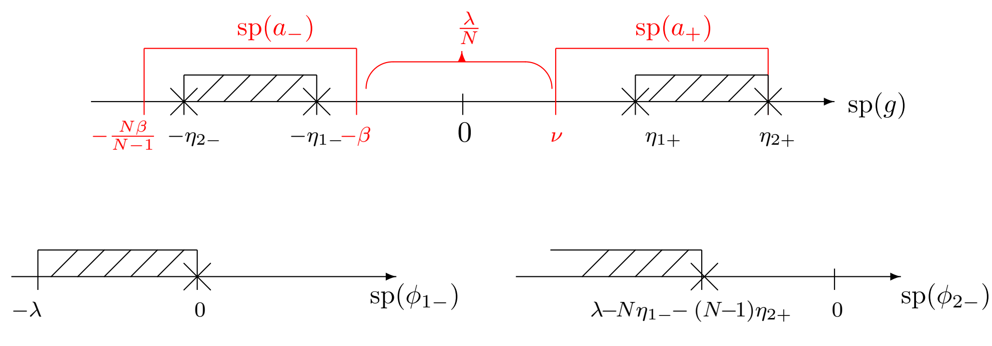


In the second case, $N\eta_{1-} + (N-1)\eta_{2+} \geq \lambda$, choosing $\nu = \frac{N-1}{N}\eta_{2+}$, it is not difficult to see that for this ν,

$$\max \mathrm{sp}(\phi_{1-}) = 0, \tag{4.118}$$

since $\eta_{2+} \in \mathrm{sp}(g_+)$.

Analogously to the previous case, the next figures illustrate the relation between this choice of ν and $\mathrm{sp}(\phi_{1-})$ and $\mathrm{sp}(\phi_{2-})$.



Now, taking this into account and Theorem 3.5 we have the following.

Theorem 4.18. *Let G be of the form (4.1) where g is given by (4.101) satisfying (4.102)-(4.107). Let us assume moreover that*

$$\inf_{\xi \in S}(|g_+(\xi)| + |g_-(\xi)|) > 0 \tag{4.119}$$

for any strip $S = \{\xi \in \mathbb{C} : -\varepsilon_2 < \mathrm{Im}(\xi) < \varepsilon_1\}$, $\varepsilon_1, \varepsilon_2 \in [\,0, +\infty\,[$ where $g_\pm$ are holomorphic.

(i) *If $N\eta_{1-} + (N-1)\eta_{2+} \leq \lambda$, then G admits an APW factorization with AP partial indices $\pm\delta$,*

$$\delta = \min\{N\eta_{1-} - (N-1)\eta_{2-},\, N(\eta_{1+} + \eta_{1-}) - \lambda\} \tag{4.120}$$

and a canonical factorization exists iff

$$\eta_{1+} + \eta_{1-} = \frac{\lambda}{N}, \qquad \textit{for} \quad N \geq 1 \tag{4.121}$$

$$\textit{or} \qquad \eta_{1-} = \frac{N-1}{N}\eta_{2-}, \qquad \textit{for} \quad N > 1. \tag{4.122}$$

(ii) *If $N\eta_{1-} + (N-1)\eta_{2+} \geq \lambda$, then G admits an APW factorization with AP partial indices $\pm\delta$,*

$$\delta = \min\{\lambda - (N-1)(\eta_{2+} + \eta_{2-}),\, N\eta_{1+} - (N-1)\eta_{2+}\} \tag{4.123}$$

and a canonical factorization exists iff, for $N > 1$,

$$\eta_{2+} + \eta_{2-} = \frac{\lambda}{N-1} \qquad or \qquad \eta_{1+} = \frac{N-1}{N}\eta_{2+}. \tag{4.124}$$

Proof. Let us start to consider the first case, where $N\eta_{1-} + (N-1)\eta_{2+} \leq \lambda$. As it was mentioned before, we can choose, in this case, $\nu = \frac{\lambda}{N} - \eta_{1-}$.

First of all, it should be remarked that $\phi_+ = (\phi_{1+}, \phi_{2+})$, given by (4.113) and (4.114), can be written in the form

$$\phi_+ = e_\delta \tilde{\phi}_+ \tag{4.125}$$

with $\delta > 0$ and $\tilde{\phi}_+ \in APW^+$. As a matter of fact, since $-\eta_{2-} \in \operatorname{sp}(g_-)$ we have

$$\min \operatorname{sp}(\phi_{1+}) = N\eta_{1-} - (N-1)\eta_{2-}\,. \tag{4.126}$$

Moreover, as $\eta_{1+} \in \operatorname{sp}(g_+)$, from (4.114), it is easy to see that

$$\min \operatorname{sp}(\phi_{2+}) = N(\eta_{1+} + \eta_{1-}) - \lambda. \tag{4.127}$$

Therefore, ϕ_+ can be written in the form (4.125) with

$$\delta = \min\{N\eta_{1-} - (N-1)\eta_{2-},\, N(\eta_{1+} + \eta_{1-}) - \lambda\}. \tag{4.128}$$

So to prove that G admits an APW factorization, from Theorem 3.5 it is enough to show that

$$\phi_- \in CP^- \quad \text{and} \quad \tilde{\phi}_+ \in CP^+. \tag{4.129}$$

One the one hand, from (4.117) we have that $0 \in \operatorname{sp}(\phi_{2-})$ so, it follows, for sufficiently big $\varepsilon_2 \in [0, +\infty[$,

$$\inf_{\xi \in \mathbb{C}^- - i\varepsilon_2} (|\phi_{1-}(\xi)| + |\phi_{2-}(\xi)|) > 0. \tag{4.130}$$

On the other hand, it is easy to see that we have, for sufficiently big $\varepsilon_1 \in [0, +\infty[$,

$$\inf_{\xi \in \mathbb{C}^+ + i\varepsilon_1} (|\tilde{\phi}_{1+}(\xi)| + |\tilde{\phi}_{2+}(\xi)|) > 0, \tag{4.131}$$

because if $\delta = N\eta_{1-} - (N-1)\eta_{2-}$ in (4.128), then from (4.125) and (4.126) we get $0 \in \mathrm{sp}(\tilde{\phi}_{1+})$. The same happens if $\delta = N(\eta_{1+} + \eta_{1-}) - \lambda$, since (4.125) and (4.127) imply that $0 \in \mathrm{sp}(\tilde{\phi}_{2+})$.

Since the Riemann-Hilbert problem $G\phi_+ = \phi_-$, where $\phi_\pm$ are given by (4.113)–(4.116), is equivalent to

$$\tilde{G}\tilde{\phi}_+ = \phi_-$$

where $\tilde{G} = e_\delta G$ and δ is given by (4.128), we conclude from Theorem 3.11, that it is enough to show

$$\inf_{\xi \in S} (|\phi_{1-}(\xi)| + |\phi_{2-}(\xi)|) > 0 \tag{4.132}$$

in a strip $S = \{\xi \in \mathbb{C} : -\varepsilon_2 < \mathrm{Im}(\xi) < \varepsilon_1\}$, for $\varepsilon_1, \varepsilon_2 \geq 0$, such that (4.130) and (4.131) hold. It should be noticed that all elements of $\tilde{G}$ are entire functions and that $\det \tilde{G} = e_\delta \in H_\infty(S)$.

When $N = 1$ we can take $\varepsilon_1 = \varepsilon_2 = 0$ because (4.129) is obviously satisfied. In fact, since ϕ_{1-} is an exponential function which does not approach zero in all complex plane, except "at infinity", and $0 \in \mathrm{sp}(\phi_{2-})$, we conclude that $\phi_- \in CP^-$ and analogously we conclude that $\tilde{\phi}_+ \in CP^+$.

For $N > 1$, since $a_\pm$ differ from $g_\pm$ by an exponential function, from (4.119) and Corollary 4.13, we get (4.132) and therefore (4.129).

So from Theorem 3.5 we conclude that G admits an APW factorization with AP partial indices $\pm\delta$, where δ is given by (4.128). Moreover, a canonical factorization exists iff $\delta = 0$, which in this case corresponds to (4.121) and (4.122).

In the second case, when $N\eta_{1-} + (N-1)\eta_{2+} \geq \lambda$, choosing $\nu = \frac{N-1}{N}\eta_{2+}$ and studying the solution to the Riemann-Hilbert problem, in the same way as it was done in the previous case, we get

$$0 \in \operatorname{sp}(\phi_{1-})$$
$$\min \operatorname{sp}(\phi_{1+}) = \lambda - (N-1)(\eta_{2+} + \eta_{2-})$$
$$\min \operatorname{sp}(\phi_{2+}) = N\eta_{1+} - (N-1)\eta_{2+}.$$

Following the same reasoning as in the previous case we conclude that an APW factorization exists with AP partial indices $\pm\delta$,

$$\delta = \min\{\lambda - (N-1)(\eta_{2+} + \eta_{2-}), N\eta_{1+} - (N-1)\eta_{2+}\}$$

and a canonical factorization exists iff (4.124) holds. ■

Remark 4.19. *This result was proved for $N = 1$ in [14] using a different approach.*

To apply these results, namely Theorem 4.18, we must be able to check if (4.119) hold which is, in general, a difficult task. Nevertheless, there are some particular cases where it is rather simple to verify it, for instance when g_+ or g_- is an exponential function. Now, we will consider some other examples where this condition is not difficult to check, starting with the case where g_- has two points in its spectrum.

Example I

Let us start to consider that the non-diagonal entry in (4.1) is given by (4.101), where g_- can be written in the form

$$g_- = c_{-2}e_{-\eta_{2-}} + c_{-1}e_{-\eta_{1-}} \quad \text{with} \quad c_{-2}, c_{-1} \in \mathbb{C} \setminus \{0\},$$

such that (4.102)–(4.107) hold.

Writing g_- in the form $g_- = e_{-\eta_{2-}}\tilde{g}_-$, where $\tilde{g}_- = c_{-2} + c_{-1}e_{\eta_{2-}-\eta_{1-}}$, it is clear that (4.119) is equivalent to

$$\inf_{\xi \in S}(|g_+(\xi)| + |\tilde{g}_-(\xi)|) > 0. \tag{4.133}$$

It is easy to see that the zeros of $\tilde{g}_-$ are given by

$$z_k = \frac{1}{\eta_{2-} - \eta_{1-}}\left(\arg\left(-\frac{c_{-2}}{c_{-1}}\right) + 2k\pi - i\log\left|\frac{c_{-2}}{c_{-1}}\right|\right), \quad k \in \mathbb{Z}. \tag{4.134}$$

So, in this case, we conclude that (4.133) and therefore (4.119), takes the form

$$\inf_{k \in \mathbb{Z}} |g_+(z_k)| > 0 \tag{4.135}$$

where z_k is given by (4.134).

To understand the meaning of condition (4.135), we will consider a particular case where g_+ has also two points in its spectrum.

Example I.1

Let us suppose in addition that g_+ has the form:

$$g_+ = c_1 e_{\eta_{1+}} + c_2 e_{\eta_{2+}} \qquad \text{with} \quad c_1, c_2 \in \mathbb{C} \setminus \{0\}\,.$$

In this case, writing $g_+ = c_2 e_{\eta_{1+}}\left(\frac{c_1}{c_2} + e_{\eta_{2+}-\eta_{1+}}\right)$ we conclude, using the results of Chapter 3, that (4.135) holds iff

$$\left|\frac{c_1}{c_2}\right|^{\eta_{2-}-\eta_{1-}} \neq \left|\frac{c_{-2}}{c_{-1}}\right|^{\eta_{2+}-\eta_{1+}}, \quad \text{if} \quad \frac{\eta_{2+}-\eta_{1+}}{\eta_{2-}-\eta_{1-}} \quad \text{is} \quad \text{irrational}; \tag{4.136}$$

$$\left(-\frac{c_1}{c_2}\right)^p \neq \left(-\frac{c_{-2}}{c_{-1}}\right)^q, \quad \text{if} \quad \frac{\eta_{2+}-\eta_{1+}}{\eta_{2-}-\eta_{1-}} \quad \text{is} \quad \text{rational} \tag{4.137}$$

with $p, q \in \mathbb{N}$ relatively prime such that $\frac{\eta_{2+}-\eta_{1+}}{\eta_{2-}-\eta_{1-}} = \frac{p}{q}$.

Thus we conclude that if (4.136) and (4.137) hold, then G admits an APW factorization and (i) and (ii) in Theorem 4.18 give us the AP partial indices and, in particular, necessary and sufficient conditions for existence of a canonical factorization of G.

In this example g is a quadrinomial, i.e., $\mathrm{sp}(g)$ consists of four different points. It may be worth remarking that another case where G is of the same form as here, with quadrinomial g, is also studied in [1] and [26] and, although $g \notin S_{\frac{\lambda}{N}}$ in that case, conditions similar to (4.136) and (4.137) are obtained for a canonical factorization of G.

Example II

To finish this section, let us now consider another example where it is also easy to see that g_+ and g_- do not approach zero simultaneously in a strip so that (4.119) holds for any S as in Theorem 4.18.

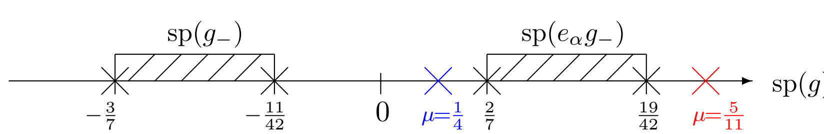
Let g be given by (4.101) with

$$g_+ = e_\alpha g_- + e_\mu, \tag{4.138}$$

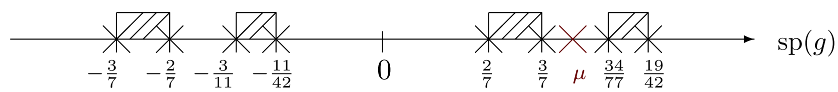
where $g_\pm$ satisfy (4.102)–(4.107), assuming $N = 2$ and $\lambda = 1$, for simplicity. Let us assume moreover $-\eta_{1-} + \eta_{2-} = \frac{1}{6}$, $\alpha, \mu > 0$ and $\mu \notin \mathrm{sp}(e_\alpha g_-)$.

There are several examples that can illustrate this case, for instance when $\eta_{2-} = \frac{3}{7}$, $\alpha = \frac{5}{7}$.

We can take, for example, $\mu = \frac{5}{11}$ or $\mu = \frac{1}{4}$ and in both cases we have that $\mu \notin \mathrm{sp}(e_\alpha g_-)$, as it is illustrated in the next figure.



If, in addition, g_- is such that $\mathrm{sp}(g_-) \subset \left[-\frac{3}{7}, -\frac{2}{7}\right] \cup \left[-\frac{3}{11}, -\frac{11}{42}\right]$, then we can take $\mu = \frac{7}{16}$, which is not a point of the spectrum of $e_\alpha g_-$ (see next figure).



Chapter 5

Appendix: The Portuguese Transformation

In this chapter, we will give a brief description of the Portuguese Transformation, which is used in Section 4.1 to compare the obtained results with the already studied ones. This method was introduced and used for the first time in [3] and a detailed exposition can be found in [7].

When we are interested in finding or proving the non-existence of an AP factorization of the matrix function G of the form

$$G = G_g^{(\lambda)} = \begin{bmatrix} e_{-\lambda} & 0 \\ g & e_{\lambda} \end{bmatrix} \quad \text{where} \quad \lambda > 0 \quad \text{and} \quad g \in AP\,, \tag{5.1}$$

an useful tool is an algorithm known as the *Portuguese Transformation*, which is a procedure of reducing the factorization problem for one matrix function to another one. By applying this transformation, eventually more than once, we may be led to a matrix function which is factorable only simultaneously with G, but for which the factorization problem has a known solution.

Part of the literature expresses things in terms of right factorizations

$(G = G_- D G_+)$, whereas the other part uses the language of left factorizations $(G = G_+ D G_-)$. Almost all papers devoted to the Portuguese Transformation deal with left factorizations; in this chapter we consider right factorizations as is consistent with the preceding parts.

Clearly, from a right AP factorization of G, one may obtain a left AP factorization of G^{-1}, taking the inverse of (5.1). Analogously, from a left AP factorization of the matrix G one can get a right factorization of G^{-1} (cf. [7], [24]). In fact, if we have

$$G = G_- D G_+ \text{ is a right AP factorization of } G$$

$$\text{iff}$$

$$G^{-1} = G_+^{-1} D^{-1} G_-^{-1} \text{ is a left } AP \text{ factorization of } G^{-1}.$$

Therefore, in what follows we focus our attention on right AP factorization and if one wants a left factorization of a given matrix G, one can do this by applying the results to the matrix G^{-1}. Through the rest of this chapter we omit the word "right" thus instead of right AP factorization of G we will simply speak of AP factorization of G.

Taking this into account, a brief description of the Portuguese Transformation will be given afterwards, but of course with some changes from that one presented in [7], which corresponds to the change from a left to a right factorization.

It was proved in [7] (see Proposition 13.4) that the problem of factorizing $G_g^{(\lambda)}$ can be reduced to the task of finding a factorization of this matrix such that

$$\mathrm{sp}(g) \subset (-\lambda, \lambda)\,, \qquad \lambda > 0. \tag{5.2}$$

Therefore, we will assume without loss of generality that (5.2) holds.

In the Portuguese Transformation, we are interested in reducing the factorization of $G_g^{(\lambda)}$ to the factorization of some other matrix function $G_{g_1}^{(\nu)}$, where $\nu < \lambda$. Towards this end, we want to construct an invertible matrix function in $(AP^+)^{2\times 2}$

$$\begin{bmatrix} f_2 & v \\ f_1 & u \end{bmatrix} \in \mathcal{G}(AP^+)^{2\times 2} \tag{5.3}$$

such that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e_{-\lambda} & 0 \\ g & e_\lambda \end{bmatrix} \begin{bmatrix} f_2 & v \\ f_1 & u \end{bmatrix} = \begin{bmatrix} e_{-\nu} & 0 \\ g_1 & e_\nu \end{bmatrix} \tag{5.4}$$

where $\nu \in (-\infty\, , \lambda)$ and

$$\begin{bmatrix} e_{-\nu} & 0 \\ g_1 & e_\nu \end{bmatrix} = G_{g_1}^{(\nu)}.$$

Equation (5.4) is equivalent to the four equations

$$v\, e_{-\lambda} = e_\nu\,, \tag{5.5}$$

$$f_2\, e_{-\lambda} = g_1\,, \tag{5.6}$$

$$u\, e_\lambda \,+\, g\, v = 0\,, \tag{5.7}$$

$$f_1\, e_{\lambda+\nu} + f_2\, e_\nu\, g = 1\,. \tag{5.8}$$

Taking (5.5)-(5.8) into account, we can write

$$\begin{bmatrix} f_2 & v \\ f_1 & u \end{bmatrix} = \begin{bmatrix} f_2 & e_{\lambda+\nu} \\ f_1 & -e_\nu g \end{bmatrix}.$$

It is not difficult to see that this matrix belongs to $(AP^+)^{2\times 2}$ if and only if

$$\mathrm{sp}(e_\nu g) \subset [0, \infty[\tag{5.9}$$

$$\nu \in [-\lambda, \infty[\tag{5.10}$$

$$f_1, f_2 \in AP^+. \tag{5.11}$$

As it was stated in the beginning of this chapter, we can consider $\nu \in (-\lambda, \lambda)$. So we simply have to demand that $\mathrm{sp}(e_\nu g) \subset [0, \infty[$ for some $\nu \in (-\lambda, \lambda)$, otherwise the Portuguese Transformation is not applicable. Then, of course we have (5.9) and (5.10). If, in addition, (5.11) is satisfied, applying determinants to (5.4), we see that the matrix (5.3) is automatically invertible in $(AP^+)^{2\times 2}$.

In summary we have:
given $g \in AP$ such that

$$\mathrm{sp}(e_\nu g) \subset [0, \infty[\quad \text{for} \quad \text{some} \quad \nu \in (-\lambda, \lambda)\,, \tag{5.12}$$

we want to find

$$f_1, f_2 \in AP^+ \quad \text{so} \quad \text{that} \quad f_1 e_{\lambda+\nu} + f_2 e_\nu g = 1 \tag{5.13}$$

and replace $G_g^{(\lambda)}$ by $G_{g_1}^{(\nu)}$ where

$$g_1 = f_2\, e_{-\lambda}. \tag{5.14}$$

So $G_g^{(\lambda)}$ is AP factorable if and only if $G_{g_1}^{(\nu)}$ is AP factorable. In the case of AP factorability, the AP indices of $G_g^{(\lambda)}$ and $G_{g_1}^{(\nu)}$ coincide. The passage from $G_g^{(\lambda)}$ to $G_{g_1}^{(\nu)}$ will be called the *Portuguese Transformation*.

The crucial step in this algorithm, is the solution of problem (5.13) under the assumption (5.12). This problem is a special corona problem. We now turn our attention to this problem.

Let $\Sigma \subset [0, \infty[$ be an additive semigroup and suppose $0 \in \Sigma$. Let us assume also $q \in APP_\Sigma$, i.e., $q \in APP$ such that $\mathrm{sp}(q) \subset \Sigma$, given by

$$q = q_0 + \sum_{j=1}^{m} q_j e_{\gamma_j}, \tag{5.15}$$

with $q_0 \neq 0$ and $0 < \gamma_1 < \cdots < \gamma_m < \eta$ with $\gamma_j \in \Sigma$ for all j. Put $b_j := -q_0^{-1} q_j$ $(j = 1, \ldots, m)$ and, for $n = (n_1, n_2, \ldots, n_m)$ in $\mathbb{Z}_+^m$, set

$$d_n := \frac{(n_1 + \cdots + n_m)!}{n_1! \ldots n_m!} b_1^{n_1} \ldots b_m^{n_m}. \tag{5.16}$$

Further, let $\gamma := (\gamma_1, \dots , \gamma_m)$ and define

$$f_2 \ := \sum_{(n,\gamma)<\eta} q_0^{-1} d_n e_{(n,\gamma)} \tag{5.17}$$

$$f_1 \ := \sum_{j=1}^{m} b_j \sum_{\eta-\gamma_j \le (n,\gamma)<\eta} d_n e_{(n,\gamma)+\gamma_j-\eta}, \tag{5.18}$$

where $(n, \gamma) := n_1\gamma_1 + \cdots + n_m\gamma_m$ and n in (5.17) and (5.18) is taken from $\mathbb{Z}_+^m$. Obviously, $\mathrm{sp}(f_2) \subset \Sigma \cap [0, \eta[$. However, $\mathrm{sp}(f_1)$ need not be contained in Σ.

In [7], it was proved the following.

Theorem 5.1. *Let $q \in APP_\Sigma$ be given by (5.15) and define $f_1, f_2 \in APP^+$ by (5.17) and (5.18). Then*

$$f_1 e_\eta + f_2 q = 1. \tag{5.19}$$

So taking $q = e_\nu g$ and $\eta = \lambda + \nu$ in (5.19) we conclude that problem (5.13) is solvable and its solution is given by f_1, f_2 of the form (5.17), (5.18), in the case where q is of the form (5.15).

To finish this chapter, it should be remarked that sometimes to study the factorization of the matrix G, it can be more convenient to study the matrix JG^*J which is factorable only simultaneously with G (cf. [7]). In fact, given $G = (a_{jk}) \in \mathcal{G}\left(L_\infty(\mathbb{R})\right)^{2\times 2}$,

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and denoting by G^T and G^* the transpose and adjoint of G

$$G^T(x) = \left(a_{kj}(x)\right), \qquad G^*(x) = (\overline{a_{kj}(x)})$$

we have that

$$G = G_- D G_+ \text{ is an } AP \text{ factorization of } G$$

iff

$$\begin{aligned} JG^*J &= (J\overline{G}_+^T J)(J\overline{D}J)(J\overline{G}_-^T J) \\ &= \widetilde{G}_- \widetilde{D} \widetilde{G}_+ \quad \text{is an } AP \text{ factorization of } JG^*J. \end{aligned}$$

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